Commutative Complex Numbers in Four Dimensions

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Abstract

Commutative complex numbers of the form $u = x + \alpha y + \beta z + \gamma t$ in 4 dimensions are studied, the variables x, y, z and t being real numbers. Four distinct types of multiplication rules for the complex bases α, β and γ are investigated, which correspond to hypercomplex entities called in this paper circular fourcomplex numbers, hyperbolic fourcomplex numbers, planar fourcomplex numbers, and polar fourcompex numbers. Exponential and trigonometric forms for the fourcomplex numbers are given in all these cases. Expressions are given for the elementary functions of the fourcomplex variables mentioned above. Relations of equality exist between the partial derivatives of the real components of the functions of fourcomplex variables. The integral of a fourcomplex function between two points is independent of the path connecting the points. The concepts of poles and residues can be introduced for the circular, planar, and polar fourcomplex numbers, for which the exponential forms depend on cyclic variables. A hypercomplex polynomial can be written as a product of linear factors for circular and planar fourcomplex numbers, and as a product of linear or quadratic factors for the hyperbolic and polar fourcomplex numbers.

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1 Introduction

A regular, two-dimensional complex number $x + iy$ can be represented geometrically by the modulus $\rho = (x^2 + y^2)^{1/2}$ and by the polar angle $\theta = \arctan(y/x)$. The modulus ρ is multiplicative and the polar angle θ is additive upon the multiplication of ordinary complex numbers.

The quaternions of Hamilton are a system of hypercomplex numbers defined in four dimensions, the multiplication being a noncommutative operation, [1] and many other hypercomplex systems are possible, [2]-[4] but these hypercomplex systems do not have all the required properties of regular, two-dimensional complex numbers which rendered possible the development of the theory of functions of a complex variable.

This paper belongs to a series of studies on commutative complex numbers in n dimensions. [5] Systems of hypercomplex numbers in 4 dimensions of the form $u = x+\alpha y+\beta z+\gamma t$ are described in this work, where the variables x, y, z and t are real numbers, for which the multiplication is both associative and commutative. The product of two fourcomplex numbers is equal to zero if both numbers are equal to zero, or if the numbers belong to certain four-dimensional hyperplanes as discussed further in this work. The fourcomplex numbers are rich enough in properties such that an exponential and trigonometric forms exist and the concepts of analytic fourcomplex function, contour integration and residue can be defined. Expressions are given for the elementary functions of fourcomplex variable. The functions $f(u)$ of fourcomplex variable defined by power series, have derivatives $\lim_{u\to u_0}[f(u)-f(u_0)]/(u-u_0)$ independent of the direction of approach of u to u_0. If the fourcomplex function $f(u)$ of the fourcomplex variable u is written in terms of the real functions $P(x,y,z,t), Q(x,y,z,t), R(x,y,z,t), S(x,y,z,t)$, then relations of equality exist between partial derivatives of the functions P, Q, R, S. The integral $\int_A^B f(u)du$ of a fourcomplex function between two points A, B is independent of the path connecting A, B.

Four distinct types of hypercomplex numbers are studied, as discussed further. In Sec. II, the multiplication rules for the complex units α, β and γ are $\alpha^2 = -1$, $\beta^2 = -1$, $\gamma^2 = 1, \alpha\beta = \beta\alpha = -\gamma$, $\alpha\gamma = \gamma\alpha = \beta$, $\beta\gamma = \gamma\beta = \alpha$. The exponential form of a fourcomplex number is $x + \alpha y + \beta z + \gamma t = \rho\exp\left[\gamma\ln\tan\psi + (1/2)\alpha(\phi+\chi) + (1/2)\beta(\phi-\chi)\right]$, where the amplitude is $\rho^4 = \left[(x+t)^2 + (y+z)^2\right]\left[(x-t)^2 + (y-z)^2\right]$, ϕ, χ are azimuthal angles,

$0 \le \phi < 2\pi, 0 \le \chi < 2\pi$, and ψ is a planar angle, $0 < \psi \le \pi/2$. The trigonometric form of a fourcomplex number is $x + \alpha y + \beta z + \gamma t = d[\cos(\psi - \pi/4) + \gamma \sin(\psi - \pi/4)] \exp[(1/2)\alpha(\phi + \chi) + (1/2)\beta(\phi - \chi)]$, where $d^2 = x^2 + y^2 + z^2 + t^2$. The amplitude ρ and $\tan\psi$ are multiplicative and the angles ϕ, χ are additive upon the multiplication of fourcomplex numbers. Since there are two cyclic variables, ϕ and χ, these fourcomplex numbers are called circular fourcomplex numbers. If $f(u)$ is an analytic fourcomplex function, then $\oint_\Gamma f(u)du/(u - u_0) = \pi[(\alpha + \beta)\ \mathrm{int}(u_{0\xi\upsilon}, \Gamma_{\xi\upsilon}) + (\alpha - \beta)\ \mathrm{int}(u_{0\tau\zeta}, \Gamma_{\tau\zeta})]\ f(u_0)$, where the functional int takes the values 0 or 1 depending on the relation between the projections of the point u_0 and of the curve Γ on certain planes. A polynomial can be written as a product of linear or quadratic factors, although the factorization may not be unique.

In Sec. III, the multiplication rules for the complex units α, β and γ are $\alpha^2 = 1,\ \beta^2 = 1, \gamma^2 = 1, \alpha\beta = \beta\alpha = \gamma,\ \alpha\gamma = \gamma\alpha = \beta,\ \beta\gamma = \gamma\beta = \alpha$. The exponential form of a fourcomplex number, which can be defined for $s = x+y+z+t > 0,\ s' = x-y+z-t > 0,\ s'' = x+y-z-t > 0,\ s''' = x-y-z+t > 0$, is $x+\alpha y+\beta z+\gamma t = \mu \exp(\alpha y_1 + \beta z_1 + \gamma t_1)$, where the amplitude is $\mu = (ss's''s''')^{1/4}$ and the arguments are $y_1 = (1/4)\ln(ss''/s's'''), z_1 = (1/4)\ln(ss'''/s's''), t_1 = (1/4)\ln(ss'/s''s''')$. Since there is no cyclic variable, these fourcomplex numbers are called hyperbolic fourcomplex numbers. The amplitude μ is multiplicative and the arguments y_1, z_1, t_1 are additive upon the multiplication of fourcomplex numbers. A polynomial can be written as a product of linear or quadratic factors, although the factorization may not be unique.

In Sec. IV, the multiplication rules for the complex units α, β and γ are $\alpha^2 = \beta,\ \beta^2 = -1,\ \gamma^2 = -\beta, \alpha\beta = \beta\alpha = \gamma,\ \alpha\gamma = \gamma\alpha = -1,\ \beta\gamma = \gamma\beta = -\alpha$. The exponential function of a fourcomplex number can be expanded in terms of the four-dimensional cosexponential functions $f_{40}(x) = 1 - x^4/4! + x^8/8! - \cdots,\ f_{41}(x) = x - x^5/5! + x^9/9! - \cdots,\ f_{42}(x) = x^2/2! - x^6/6! + x^{10}/10! - \cdots,\ f_{43}(x) = x^3/3! - x^7/7! + x^{11}/11! - \cdots$. Expressions are obtained for the four-dimensional cosexponential functions in terms of elementary functions. The exponential form of a fourcomplex number is $x + \alpha y + \beta z + \gamma t = \rho \exp\left\{(1/2)(\alpha - \gamma)\ln\tan\psi + (1/2)[\beta + (\alpha+\gamma)/\sqrt{2}]\phi - (1/2)[\beta - (\alpha+\gamma)/\sqrt{2}]\chi\right\}$, where the amplitude is $\rho^4 = \left\{\left[x + (y-t)/\sqrt{2}\right]^2 + \left[z + (y+t)/\sqrt{2}\right]^2\right\}\left\{\left[x - (y-t)/\sqrt{2}\right]^2 + \left[z - (y+t)/\sqrt{2}\right]^2\right\}$, ϕ, χ are azimuthal angles, $0 \le \phi < 2\pi, 0 \le \chi < 2\pi$, and ψ is a planar angle, $0 \le \psi \le \pi/2$.

The trigonometric form of a fourcomplex number is $x+\alpha y+\beta z+\gamma t = d\,[\cos(\psi-\pi/4)$ $+(1/\sqrt{2})(\alpha-\gamma)\sin(\psi-\pi/4)\big]\exp\left\{(1/2)[\beta+(\alpha+\gamma)/\sqrt{2}]\phi-(1/2)[\beta-(\alpha+\gamma)/\sqrt{2}]\chi\right\}$, where $d^2=x^2+y^2+z^2+t^2$. The amplitude ρ and $\tan\psi$ are multiplicative and the angles ϕ,χ are additive upon the multiplication of fourcomplex numbers. There are two cyclic variables, ϕ and χ, so that these fourcomplex numbers are also of a circular type. In order to distinguish them from the circular hypercomplex numbers, these are called planar fourcomplex numbers. If $f(u)$ is an analytic fourcomplex function, then $\oint_\Gamma f(u)du/(u-u_0) = \pi\left[\left(\beta+(\alpha+\gamma)/\sqrt{2}\right)\ \mathrm{int}(u_{0\xi\upsilon},\Gamma_{\xi\upsilon})\ +\left(\beta-(\alpha+\gamma)/\sqrt{2}\right)\ \mathrm{int}(u_{0\tau\zeta},\Gamma_{\tau\zeta})\right]\ f(u_0)$, where the functional int takes the values 0 or 1 depending on the relation between the projections of the point u_0 and of the curve Γ on certain planes. A polynomial can be written as a product of linear or quadratic factors, although the factorization may not be unique. The fourcomplex numbers described in this section are a particular case for $n=4$ of the planar hypercomplex numbers in n dimensions.[5],[6]

In Sec. V, the multiplication rules for the complex units α,β and γ are $\alpha^2=\beta,\ \beta^2=1,\ \gamma^2=\beta, \alpha\beta=\beta\alpha=\gamma,\ \alpha\gamma=\gamma\alpha=1,\ \beta\gamma=\gamma\beta=\alpha$. The product of two fourcomplex numbers is equal to zero if both numbers are equal to zero, or if the numbers belong to certain four-dimensional hyperplanes described further in this work. The exponential function of a fourcomplex number can be expanded in terms of the four-dimensional cosexponential functions $g_{40}(x)=1+x^4/4!+x^8/8!+\cdots,\ g_{41}(x)=x+x^5/5!+x^9/9!+\cdots,\ g_{42}(x)=x^2/2!+x^6/6!+x^{10}/10!+\cdots,\ \ g_{43}(x)=x^3/3!+x^7/7!+x^{11}/11!+\cdots$. Addition theorems and other relations are obtained for these four-dimensional cosexponential functions. The exponential form of a fourcomplex number, which can be defined for $x+y+z+t>0, x-y+z-t>0$, is $u=\rho\exp\left[(1/4)(\alpha+\beta+\gamma)\ln(\sqrt{2}/\tan\theta_+)-(1/4)(\alpha-\beta+\gamma)\ln(\sqrt{2}/\tan\theta_-)+(\alpha-\gamma)\phi/2\right]$, where $\rho=(\mu_+\mu_-)^{1/2}, \mu_+^2=(x-z)^2+(y-t)^2, \mu_-^2=(x+z)^2-(y+t)^2, e_+=(1+\alpha+\beta+\gamma)/4, e_-=(1-\alpha+\beta-\gamma)/4$, $e_1=(1-\beta)/2, \tilde{e}_1=(\alpha-\gamma)/2$, the polar angles are $\tan\theta_+=\sqrt{2}\mu_+/v_+, \tan\theta_-=\sqrt{2}\mu_+/v_-$, $0\le\theta_+\le\pi, 0\le\theta_-\le\pi$, and the azimuthal angle ϕ is defined by the relations $x-y=\mu_+\cos\phi,\ \ z-t=\mu_+\sin\phi,\ 0\le\phi<2\pi$. The trigonometric form of the fourcomplex number u is $u=d\sqrt{2}\ \left(1+1/\tan^2\theta_++1/\tan^2\theta_-\right)^{-1/2}\left\{e_1+e_+\sqrt{2}/\tan\theta_++e_-\sqrt{2}/\tan\theta_-\right\}$ $\exp\left[\tilde{e}_1\phi\right]$. The amplitude ρ and $\tan\theta_+/\sqrt{2}, \tan\theta_-/\sqrt{2}$, are multiplicative, and the azimuthal angle ϕ is additive upon the multiplication of fourcomplex numbers. There is only one

cyclic variable, ϕ, and there are two axes v_+, v_- which play an important role in the description of these numbers, so that these hypercomplex numbers are called polar fourcomplex numbers. If $f(u)$ is an analytic fourcomplex function, then $\oint_\Gamma f(u)du/(u-u_0) = \pi(\beta-\gamma)\ \mathrm{int}(u_{0\xi\upsilon}, \Gamma_{\xi\upsilon})f(u_0)$, where the functional int takes the values 0 or 1 depending on the relation between the projections of the point u_0 and of the curve Γ on certain planes. A polynomial can be written as a product of linear or quadratic factors, although the factorization may not be unique. The fourcomplex numbers described in this section are a particular case for $n=4$ of the polar hypercomplex numbers in n dimensions.[5],[6]

2 Circular Complex Numbers in Four Dimensions

2.1 Operations with circular fourcomplex numbers

A circular fourcomplex number is determined by its four components (x, y, z, t). The sum of the circular fourcomplex numbers (x, y, z, t) and (x', y', z', t') is the circular fourcomplex number $(x+x', y+y', z+z', t+t')$. The product of the circular fourcomplex numbers (x, y, z, t) and (x', y', z', t') is defined in this work to be the circular fourcomplex number $(xx'-yy'-zz'+tt', xy'+yx'+zt'+tz', xz'+zx'+yt'+ty', xt'+tx'-yz'-zy')$.

Circular fourcomplex numbers and their operations can be represented by writing the circular fourcomplex number (x, y, z, t) as $u = x+\alpha y+\beta z+\gamma t$, where α, β and γ are bases for which the multiplication rules are

$$\alpha^2=-1,\ \beta^2=-1,\ \gamma^2=1, \alpha\beta=\beta\alpha=-\gamma,\ \alpha\gamma=\gamma\alpha=\beta,\ \beta\gamma=\gamma\beta=\alpha. \tag{1}$$

Two circular fourcomplex numbers $u = x+\alpha y+\beta z+\gamma t, u' = x'+\alpha y'+\beta z'+\gamma t'$ are equal, $u = u'$, if and only if $x = x', y = y', z = z', t = t'$. If $u = x+\alpha y+\beta z+\gamma t, u' = x'+\alpha y'+\beta z'+\gamma t'$ are circular fourcomplex numbers, the sum $u+u'$ and the product uu' defined above can be obtained by applying the usual algebraic rules to the sum $(x+\alpha y+\beta z+\gamma t)+(x'+\alpha y'+\beta z'+\gamma t')$ and to the product $(x+\alpha y+\beta z+\gamma t)(x'+\alpha y'+\beta z'+\gamma t')$, and grouping of the resulting terms,

$$u+u' = x+x'+\alpha(y+y')+\beta(z+z')+\gamma(t+t'), \tag{2}$$

$$uu' = xx' - yy' - zz' + tt' + \alpha(xy' + yx' + zt' + tz') + \beta(xz' + zx' + yt' + ty')$$
$$+\gamma(xt' + tx' - yz' - zy') \tag{3}$$

If u, u', u'' are circular fourcomplex numbers, the multiplication is associative

$$(uu')u'' = u(u'u'') \tag{4}$$

and commutative

$$uu' = u'u, \tag{5}$$

as can be checked through direct calculation. The circular fourcomplex zero is $0 + \alpha \cdot 0 + \beta \cdot 0 + \gamma \cdot 0$, denoted simply 0, and the circular fourcomplex unity is $1 + \alpha \cdot 0 + \beta \cdot 0 + \gamma \cdot 0$, denoted simply 1.

The inverse of the circular fourcomplex number $u = x + \alpha y + \beta z + \gamma t$ is a circular fourcomplex number $u' = x' + \alpha y' + \beta z' + \gamma t'$ having the property that

$$uu' = 1. \tag{6}$$

Written on components, the condition, Eq. (6), is

$$\begin{aligned} xx' - yy' - zz' + tt' &= 1, \\ yx' + xy' + tz' + zt' &= 0, \\ zx' + ty' + xz' + yt' &= 0, \\ tx' - zy' - yz' + xt' &= 0. \end{aligned} \tag{7}$$

The system (7) has the solution

$$x' = \frac{x(x^2 + y^2 + z^2 - t^2) - 2yzt}{\rho^4}, \tag{8}$$

$$y' = \frac{y(-x^2 - y^2 + z^2 - t^2) + 2xzt}{\rho^4}, \tag{9}$$

$$z' = \frac{z(-x^2 + y^2 - z^2 - t^2) + 2xyt}{\rho^4}, \tag{10}$$

$$t' = \frac{t(-x^2 + y^2 + z^2 + t^2) - 2xyz}{\rho^4}, \tag{11}$$

provided that $\rho \neq 0$, where

$$\rho^4 = x^4 + y^4 + z^4 + t^4 + 2(x^2y^2 + x^2z^2 - x^2t^2 - y^2z^2 + y^2t^2 + z^2t^2) - 8xyzt. \tag{12}$$

The quantity ρ will be called amplitude of the circular fourcomplex number $x+\alpha y+\beta z+\gamma t$. Since

$$\rho^4 = \rho_+^2 \rho_-^2, \tag{13}$$

where

$$\rho_+^2 = (x+t)^2 + (y+z)^2,\ \rho_-^2 = (x-t)^2 + (y-z)^2, \tag{14}$$

a circular fourcomplex number $u = x+\alpha y+\beta z+\gamma t$ has an inverse, unless

$$x+t=0,\ y+z=0, \tag{15}$$

or

$$x-t=0,\ y-z=0. \tag{16}$$

Because of conditions (15)-(16) these 2-dimensional surfaces will be called nodal hyperplanes. It can be shown that if $uu'=0$ then either $u=0$, or $u'=0$, or one of the circular fourcomplex numbers is of the form $x+\alpha y+\beta y+\gamma x$ and the other of the form $x'+\alpha y'-\beta y'-\gamma x'$.

2.2 Geometric representation of circular fourcomplex numbers

The circular fourcomplex number $x+\alpha y+\beta z+\gamma t$ can be represented by the point A of coordinates (x,y,z,t). If O is the origin of the four-dimensional space x,y,z,t, the distance from A to the origin O can be taken as

$$d^2 = x^2+y^2+z^2+t^2. \tag{17}$$

The distance d will be called modulus of the circular fourcomplex number $x+\alpha y+\beta z+\gamma t$, $d=|u|$. The orientation in the four-dimensional space of the line OA can be specified with the aid of three angles ϕ,χ,ψ defined with respect to the rotated system of axes

$$\xi = \frac{x+t}{\sqrt{2}},\ \tau = \frac{x-t}{\sqrt{2}},\ \upsilon = \frac{y+z}{\sqrt{2}},\ \zeta = \frac{y-z}{\sqrt{2}}. \tag{18}$$

The variables ξ,υ,τ,ζ will be called canonical circular fourcomplex variables. The use of the rotated axes ξ,υ,τ,ζ for the definition of the angles ϕ,χ,ψ is convenient for the expression

of the circular fourcomplex numbers in exponential and trigonometric forms, as it will be discussed further. The angle ϕ is the angle between the projection of A in the plane ξ, υ and the $O\xi$ axis, $0 \leq \phi < 2\pi$, χ is the angle between the projection of A in the plane τ, ζ and the $O\tau$ axis, $0 \leq \chi < 2\pi$, and ψ is the angle between the line OA and the plane $\tau O\zeta$, $0 \leq \psi \leq \pi/2$, as shown in Fig. 1. The angles ϕ and χ will be called azimuthal angles, the angle ψ will be called planar angle . The fact that $0 \leq \psi \leq \pi/2$ means that ψ has the same sign on both faces of the two-dimensional hyperplane $\upsilon O\zeta$. The components of the point A in terms of the distance d and the angles ϕ, χ, ψ are thus

$$\frac{x+t}{\sqrt{2}} = d\cos\phi\sin\psi, \tag{19}$$

$$\frac{x-t}{\sqrt{2}} = d\cos\chi\cos\psi, \tag{20}$$

$$\frac{y+z}{\sqrt{2}} = d\sin\phi\sin\psi, \tag{21}$$

$$\frac{y-z}{\sqrt{2}} = d\sin\chi\cos\psi. \tag{22}$$

It can be checked that $\rho_+ = \sqrt{2}d\sin\psi, \rho_- = \sqrt{2}d\cos\psi$. The coordinates x, y, z, t in terms of the variables d, ϕ, χ, ψ are

$$x = \frac{d}{\sqrt{2}}(\cos\psi\cos\chi + \sin\psi\cos\phi), \tag{23}$$

$$y = \frac{d}{\sqrt{2}}(\cos\psi\sin\chi + \sin\psi\sin\phi), \tag{24}$$

$$z = \frac{d}{\sqrt{2}}(-\cos\psi\sin\chi + \sin\psi\sin\phi), \tag{25}$$

$$t = \frac{d}{\sqrt{2}}(-\cos\psi\cos\chi + \sin\psi\cos\phi). \tag{26}$$

The angles ϕ, χ, ψ can be expressed in terms of the coordinates x, y, z, t as

$$\sin\phi = (y+z)/\rho_+, \; \cos\phi = (x+t)/\rho_+, \tag{27}$$

$$\sin\chi = (y-z)/\rho_-, \; \cos\chi = (x-t)/\rho_-, \tag{28}$$

$$\tan\psi = \rho_+/\rho_-. \tag{29}$$

The nodal hyperplanes are $\xi O \upsilon$, for which $\tau = 0, \zeta = 0$, and $\tau O \zeta$, for which $\xi = 0, \upsilon = 0$. For points in the nodal hyperplane $\xi O \upsilon$ the planar angle is $\psi = \pi/2$, for points in the nodal hyperplane $\tau O \zeta$ the planar angle is $\psi = 0$.

It can be shown that if $u_1 = x_1 + \alpha y_1 + \beta z_1 + \gamma t_1, u_2 = x_2 + \alpha y_2 + \beta z_2 + \gamma t_2$ are circular fourcomplex numbers of amplitudes and angles $\rho_1, \phi_1, \chi_1, \psi_1$ and respectively $\rho_2, \phi_2, \chi_2, \psi_2$, then the amplitude ρ and the angles ϕ, χ, ψ of the product circular fourcomplex number $u_1 u_2$ are

$$\rho = \rho_1 \rho_2, \tag{30}$$

$$\phi = \phi_1 + \phi_2,\ \chi = \chi_1 + \chi_2,\ \tan\psi = \tan\psi_1 \tan\psi_2. \tag{31}$$

The relations (30)-(31) are consequences of the definitions (12)-(14), (27)-(29) and of the identities

$$\begin{aligned}
&[(x_1x_2 - y_1y_2 - z_1z_2 + t_1t_2) + (x_1t_2 + t_1x_2 - y_1z_2 - z_1y_2)]^2 \\
&\quad+[(x_1y_2 + y_1x_2 + z_1t_2 + t_1z_2) + (x_1z_2 + z_1x_2 + y_1t_2 + t_1y_2)]^2 \\
&\quad= [(x_1 + t_1)^2 + (y_1 + z_1)^2][(x_2 + t_2)^2 + (y_2 + z_2)^2],
\end{aligned} \tag{32}$$

$$\begin{aligned}
&[(x_1x_2 - y_1y_2 - z_1z_2 + t_1t_2) - (x_1t_2 + t_1x_2 - y_1z_2 - z_1y_2)]^2 \\
&\quad+[(x_1y_2 + y_1x_2 + z_1t_2 + t_1z_2) - (x_1z_2 + z_1x_2 + y_1t_2 + t_1y_2)]^2 \\
&\quad= [(x_1 - t_1)^2 + (y_1 - z_1)^2][(x_2 - t_2)^2 + (y_2 - z_2)^2],
\end{aligned} \tag{33}$$

$$\begin{aligned}
&(x_1x_2 - y_1y_2 - z_1z_2 + t_1t_2) + (x_1t_2 + t_1x_2 - y_1z_2 - z_1y_2) \\
&\quad= (x_1 + t_1)(x_2 + t_2) - (y_1 + z_1)(y_2 + z_2),
\end{aligned} \tag{34}$$

$$\begin{aligned}
&(x_1x_2 - y_1y_2 - z_1z_2 + t_1t_2) - (x_1t_2 + t_1x_2 - y_1z_2 - z_1y_2) \\
&\quad= (x_1 - t_1)(x_2 - t_2) - (y_1 - z_1)(y_2 - z_2),
\end{aligned} \tag{35}$$

$$\begin{aligned}
&(x_1y_2 + y_1x_2 + z_1t_2 + t_1z_2) + (x_1z_2 + z_1x_2 + y_1t_2 + t_1y_2) \\
&\quad= (y_1 + z_1)(x_2 + t_2) + (y_2 + z_2)(x_2 + t_2),
\end{aligned} \tag{36}$$

$$\begin{aligned}
&(x_1y_2 + y_1x_2 + z_1t_2 + t_1z_2) - (x_1z_2 + z_1x_2 + y_1t_2 + t_1y_2) \\
&\quad= (y_1 - z_1)(x_2 - t_2) + (y_2 - z_2)(x_2 - t_2).
\end{aligned} \tag{37}$$

The identities (32) and (33) can also be written as

$$\rho_+^2 = \rho_{1+}\rho_{2+}, \tag{38}$$

$$\rho_-^2 = \rho_{1-}\rho_{2-}, \tag{39}$$

where

$$\rho_{j+}^2 = (x_j + t_j)^2 + (y_j + z_j)^2,\ \rho_{j-}^2 = (x_j - t_j)^2 + (y_j - z_j)^2,\ j = 1, 2. \tag{40}$$

The fact that the amplitude of the product is equal to the product of the amplitudes, as written in Eq. (30), can be demonstrated also by using a representation of the multiplication of the circular fourcomplex numbers by matrices, in which the circular fourcomplex number $u = x + \alpha y + \beta z + \gamma t$ is represented by the matrix

$$A = \begin{pmatrix} x & y & z & t \\ -y & x & t & -z \\ -z & t & x & -y \\ t & z & y & x \end{pmatrix}. \tag{41}$$

The product $u = x + \alpha y + \beta z + \gamma t$ of the circular fourcomplex numbers $u_1 = x_1 + \alpha y_1 + \beta z_1 + \gamma t_1, u_2 = x_2 + \alpha y_2 + \beta z_2 + \gamma t_2$, can be represented by the matrix multiplication

$$A = A_1 A_2. \tag{42}$$

It can be checked that the determinant $\det(A)$ of the matrix A is

$$\det A = \rho^4. \tag{43}$$

The identity (30) is then a consequence of the fact the determinant of the product of matrices is equal to the product of the determinants of the factor matrices.

2.3 The exponential and trigonometric forms of circular fourcomplex numbers

The exponential function of the fourcomplex variable u is defined by the series

$$\exp u = 1 + u + u^2/2! + u^3/3! + \cdots. \tag{44}$$

It can be checked by direct multiplication of the series that

$$\exp(u + u') = \exp u \cdot \exp u'. \tag{45}$$

These relations have the same form for all hypercomplex systems discussed in this work.

If $u = x + \alpha y + \beta z + \gamma t$, then $\exp u$ can be calculated as $\exp u = \exp x \cdot \exp(\alpha y) \cdot \exp(\beta z) \cdot \exp(\gamma t)$. According to Eq. (1),

$$\alpha^{2m} = (-1)^m, \alpha^{2m+1} = (-1)^m \alpha, \beta^{2m} = (-1)^m, \beta^{2m+1} = (-1)^m \beta, \gamma^m = 1, \tag{46}$$

where m is a natural number, so that $\exp(\alpha y)$, $\exp(\beta z)$ and $\exp(\gamma t)$ can be written as

$$\exp(\alpha y) = \cos y + \alpha \sin y,\ \exp(\beta z) = \cos z + \beta \sin z, \tag{47}$$

and

$$\exp(\gamma t) = \cosh t + \gamma \sinh t. \tag{48}$$

From Eqs. (47)-(48) it can be inferred that

$$\begin{aligned} (\cos y + \alpha \sin y)^m &= \cos my + \alpha \sin my, \\ (\cos z + \beta \sin z)^m &= \cos mz + \beta \sin mz, \\ (\cosh t + \gamma \sinh t)^m &= \cosh mt + \gamma \sinh mt. \end{aligned} \tag{49}$$

Any circular fourcomplex number $u = x + \alpha y + \beta z + \gamma t$ can be writen in the form

$$x + \alpha y + \beta z + \gamma t = e^{x_1 + \alpha y_1 + \beta z_1 + \gamma t_1}. \tag{50}$$

The expressions of x_1, y_1, z_1, t_1 as functions of x, y, z, t can be obtained by developing $e^{\alpha y_1}, e^{\beta z_1}$ and $e^{\gamma t_1}$ with the aid of Eqs. (47) and (48), by multiplying these expressions and separating the hypercomplex components,

$$x = e^{x_1}(\cos y_1 \cos z_1 \cosh t_1 - \sin y_1 \sin z_1 \sinh t_1), \tag{51}$$

$$y = e^{x_1}(\sin y_1 \cos z_1 \cosh t_1 + \cos y_1 \sin z_1 \sinh t_1), \tag{52}$$

$$z = e^{x_1}(\cos y_1 \sin z_1 \cosh t_1 + \sin y_1 \cos z_1 \sinh t_1), \tag{53}$$

$$t = e^{x_1}(-\sin y_1 \sin z_1 \cosh t_1 + \cos y_1 \cos z_1 \sinh t_1), \tag{54}$$

From Eqs. (51)-(54) it can be shown by direct calculation that

$$x^2+y^2+z^2+t^2=e^{2x_1}\cosh 2t_1, \tag{55}$$

$$2(xt+yz)=e^{2x_1}\sinh 2t_1, \tag{56}$$

so that

$$e^{4x_1}=(x^2+y^2+z^2+t^2)^2-4(xt+yz)^2. \tag{57}$$

By comparing the expression in the right-hand side of Eq. (57) with the expression of ρ, Eq. (13), it can be seen that

$$e^{x_1}=\rho. \tag{58}$$

The variable t_1 is then given by

$$\cosh 2t_1=\frac{d^2}{\rho^2},\ \sinh 2t_1=\frac{2(xt+yz)}{\rho^2}. \tag{59}$$

From the fact that $\rho^4=d^4-4(xt+yz)^2$ it follows that $d^2/\rho^2\geq 1$, so that Eq. (59) has always a real solution, and $t_1=0$ for $xt+yz=0$. It can be shown similarly that

$$\cos 2y_1=\frac{x^2-y^2+z^2-t^2}{\rho^2},\ \sin 2y_1=\frac{2(xy-zt)}{\rho^2}, \tag{60}$$

$$\cos 2z_1=\frac{x^2+y^2-z^2-t^2}{\rho^2},\ \sin 2z_1=\frac{2(xz-yt)}{\rho^2}. \tag{61}$$

It can be shown that $(x^2-y^2+z^2-t^2)^2\leq\rho^4$, the equality taking place for $xy=zt$, and $(x^2+y^2-z^2-t^2)^2\leq\rho^4$, the equality taking place for $xz=yt$, so that Eqs. (60) and Eqs. (61) have always real solutions.

The variables

$$y_1=\frac{1}{2}\arcsin\frac{2(xy-zt)}{\rho^2},\ z_1=\frac{1}{2}\arcsin\frac{2(xz-yt)}{\rho^2},\ t_1=\frac{1}{2}\mathrm{argsinh}\frac{2(xt+yz)}{\rho^2} \tag{62}$$

are additive upon the multiplication of circular fourcomplex numbers, as can be seen from the identities

$$\begin{aligned}&(xx'-yy'-zz'+tt')(xy'+yx'+zt'+tz')\\&\quad-(xz'+zx'+yt'+ty')(xt'-yz'-zy'+tx')\\&\quad=(xy-zt)(x'^2-y'^2+z'^2-t'^2)+(x^2-y^2+z^2-t^2)(x'y'-z't'),\end{aligned} \tag{63}$$

$$
\begin{aligned}
&(xx' - yy' - zz' + tt')(xz' + zx' + yt' + ty')\\
&\quad -(xy' + yx' + zt' + tz')(xt' - yz' - zy' + tx')\\
&\quad = (xz - yt)(x'^2 + y'^2 - z'^2 - t'^2) + (x^2 + y^2 - z^2 - t^2)(x'z' - y't'),
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
&(xx' - yy' - zz' + tt')(xt' - yz' - zy' + tx')\\
&\quad +(xy' + yx' + zt' + tz')(xz' + zx' + yt' + ty')\\
&\quad = (xt + yz)(x'^2 + y'^2 + z'^2 + t'^2) + (x^2 + y^2 + z^2 + t^2)(x't' + y'z').
\end{aligned} \tag{65}
$$

The expressions appearing in Eqs. (59)-(61) can be calculated in terms of the angles ϕ, χ, ψ with the aid of Eqs. (23)-(26) as

$$
\frac{d^2}{\rho^2} = \frac{1}{\sin 2\psi},\ \frac{2(xt + yz)}{\rho^2} = -\frac{1}{\tan 2\psi}, \tag{66}
$$

$$
\frac{x^2 - y^2 + z^2 - t^2}{\rho^2} = \cos(\phi + \chi),\ \frac{2(xy - zt)}{\rho^2} = \sin(\phi + \chi), \tag{67}
$$

$$
\frac{x^2 + y^2 - z^2 - t^2}{\rho^2} = \cos(\phi - \chi),\ \frac{2(xz - yt)}{\rho^2} = \sin(\phi - \chi). \tag{68}
$$

Then from Eqs. (59)-(61) and (66)-(68) it results that

$$
y_1 = \frac{\phi + \chi}{2},\ z_1 = \frac{\phi - \chi}{2},\ t_1 = \frac{1}{2}\ln\tan\psi, \tag{69}
$$

so that the circular fourcomplex number u, Eq. (50), can be written as

$$
u = \rho \exp\left[\alpha\frac{\phi + \chi}{2} + \beta\frac{\phi - \chi}{2} + \gamma\frac{1}{2}\ln\tan\psi\right]. \tag{70}
$$

In Eq. (70) the circular fourcomplex number $u = x + \alpha y + \beta z + \gamma t$ is written as the product of the amplitude ρ and of an exponential function, and therefore this form of u will be called the exponential form of the circular fourcomplex number. It can be checked that

$$
\exp\left(\frac{\alpha + \beta}{2}\phi\right) = \frac{1 - \gamma}{2} + \frac{1 + \gamma}{2}\cos\phi + \frac{\alpha + \beta}{2}\sin\phi, \tag{71}
$$

$$
\exp\left(\frac{\alpha - \beta}{2}\chi\right) = \frac{1 + \gamma}{2} + \frac{1 - \gamma}{2}\cos\chi + \frac{\alpha - \beta}{2}\sin\phi, \tag{72}
$$

which shows that $e^{(\alpha+\beta)\phi/2}$ and $e^{(\alpha-\beta)\chi/2}$ are periodic functions of ϕ and respectively χ, with period 2π.

The relations between the variables y_1, z_1, t_1 and the angles ϕ, χ, ψ can be obtained alternatively by substituting in Eqs. (51)-(54) the expression $e^{x_1} = d/(\cosh 2t_1)^{1/2}$, Eq. (55), and summing and subtracting of the relations,

$$\frac{x+t}{\sqrt{2}} = d\cos(y_1+z_1)\sin(\eta+\pi/4), \tag{73}$$

$$\frac{x-t}{\sqrt{2}} = d\cos(y_1-z_1)\cos(\eta+\pi/4), \tag{74}$$

$$\frac{y+z}{\sqrt{2}} = d\sin(y_1+z_1)\sin(\eta+\pi/4), \tag{75}$$

$$\frac{y-z}{\sqrt{2}} = d\sin(y_1-z_1)\cos(\eta+\pi/4), \tag{76}$$

where the variable η is defined by the relations

$$\frac{\cosh t_1}{(\cosh 2t_1)^{1/2}} = \cos\eta,\ \frac{\sinh t_1}{(\cosh 2t_1)^{1/2}} = \sin\eta, \tag{77}$$

and when $-\infty < t_1 < \infty$, the range of the variable η is $-\pi/4 \le \eta \le \pi/4$. The comparison of Eqs. (19)-(22) and (73)-(76) shows that

$$\phi = y_1 + z_1,\ \chi = y_1 - z_1,\ \psi = \eta + \pi/4. \tag{78}$$

It can be shown with the aid of Eq. (48) that

$$\exp\left(\frac{1}{2}\gamma\ln\tan\psi\right) = \frac{1}{(\sin 2\psi)^{1/2}}\left[\cos(\psi-\pi/4)+\gamma\sin(\psi-\pi/4)\right]. \tag{79}$$

The circular fourcomplex number u, Eq. (70), can then be written equivalently as

$$u = d\{\cos(\psi-\pi/4)+\gamma\sin(\psi-\pi/4)\}\exp\left[\alpha\frac{\phi+\chi}{2}+\beta\frac{\phi-\chi}{2}\right]. \tag{80}$$

In Eq. (80), the circular fourcomplex number $u = x+\alpha y+\beta z+\gamma t$ is written as the product of the modulus d and of factors depending on the geometric angles ϕ, χ and ψ, and this form will be called the trigonometric form of the circular fourcomplex number.

If u_1, u_2 are circular fourcomplex numbers of moduli and angles $d_1, \phi_1, \chi_1, \psi_1$ and respectively $d_2, \phi_2, \chi_2, \psi_2$, the product of the planar factors can be calculated to be

$$\begin{aligned}&[\cos(\psi_1-\pi/4)+\gamma\sin(\psi_1-\pi/4)][\cos(\psi_2-\pi/4)+\gamma\sin(\psi_2-\pi/4)]\\ &\quad = [\cos(\psi_1-\psi_2)-\gamma\cos(\psi_1+\psi_2)].\end{aligned} \tag{81}$$

The right-hand side of Eq. (81) can be written as

$$\begin{aligned}&\cos(\psi_1-\psi_2)-\gamma\cos(\psi_1+\psi_2)\\&\quad=[2(\cos^2\psi_1\cos^2\psi_2+\sin^2\psi_1\sin^2\psi_2)]^{1/2}[\cos(\psi-\pi/4)+\gamma\sin(\psi-\pi/4)],\end{aligned} \tag{82}$$

where the angle ψ, determined by the condition that

$$\tan(\psi-\pi/4)=-\cos(\psi_1+\psi_2)/\cos(\psi_1-\psi_2) \tag{83}$$

is given by $\tan\psi=\tan\psi_1\tan\psi_2$, which is consistent with Eq. (31). It can be checked that the modulus d of the product u_1u_2 is

$$d=\sqrt{2}d_1d_2\left(\cos^2\psi_1\cos^2\psi_2+\sin^2\psi_1\sin^2\psi_2\right)^{1/2}. \tag{84}$$

2.4 Elementary functions of a circular fourcomplex variable

The logarithm u_1 of the circular fourcomplex number u, $u_1=\ln u$, can be defined as the solution of the equation

$$u=e^{u_1}, \tag{85}$$

written explicitly previously in Eq. (50), for u_1 as a function of u. From Eq. (70) it results that

$$\ln u=\ln\rho+\frac{1}{2}\gamma\ln\tan\psi+\alpha\frac{\phi+\chi}{2}+\beta\frac{\phi-\chi}{2}. \tag{86}$$

It can be inferred from Eqs. (30) and (31) that

$$\ln(uu')=\ln u+\ln u', \tag{87}$$

up to multiples of $\pi(\alpha+\beta)$ and $\pi(\alpha-\beta)$.

The power function u^n can be defined for real values of n as

$$u^m=e^{m\ln u}. \tag{88}$$

The power function is multivalued unless n is an integer. For integer n, it can be inferred from Eq. (87) that

$$(uu')^m=u^n\,u'^m. \tag{89}$$

If, for example, $m=2$, it can be checked with the aid of Eq. (80) that Eq. (88) gives indeed $(x+\alpha y+\beta z+\gamma t)^2 = x^2-y^2-z^2+t^2+2\alpha(xy+zt)+2\beta(xz+yt)+2\gamma(2xt-yz)$.

The trigonometric functions of the fourcomplex variable u are defined by the series

$$\cos u = 1-u^2/2!+u^4/4!+\cdots, \tag{90}$$

$$\sin u = u-u^3/3!+u^5/5!+\cdots. \tag{91}$$

It can be checked by series multiplication that the usual addition theorems hold also for the circular fourcomplex numbers u, u',

$$\cos(u+u') = \cos u\cos u' - \sin u\sin u', \tag{92}$$

$$\sin(u+u') = \sin u\cos u' + \cos u\sin u'. \tag{93}$$

These relations have the same form for all systems of hypercomplex numbers discussed in this work. The cosine and sine functions of the hypercomplex variables $\alpha y, \beta z$ and γt can be expressed as

$$\cos\alpha y = \cosh y,\ \sin\alpha y = \alpha\sinh y, \tag{94}$$

$$\cos\beta y = \cosh y,\ \sin\beta y = \beta\sinh y, \tag{95}$$

$$\cos\gamma y = \cos y,\ \sin\gamma y = \gamma\sin y. \tag{96}$$

The cosine and sine functions of a circular fourcomplex number $x+\alpha y+\beta z+\gamma t$ can then be expressed in terms of elementary functions with the aid of the addition theorems Eqs. (92), (93) and of the expressions in Eqs. (94)-(96).

The hyperbolic functions of the fourcomplex variable u are defined by the series

$$\cosh u = 1+u^2/2!+u^4/4!+\cdots, \tag{97}$$

$$\sinh u = u+u^3/3!+u^5/5!+\cdots. \tag{98}$$

It can be checked by series multiplication that the usual addition theorems hold also for the circular fourcomplex numbers u, u',

$$\cosh(u+u') = \cosh u\cosh u' + \sinh u\sinh u', \tag{99}$$

$$\sinh(u+u') = \sinh u \cosh u' + \cosh u \sinh u'. \tag{100}$$

These relations have the same form for all systems of hypercomplex numbers discussed in this work. The hyperbolic cosine and sine functions of the hypercomplex variables $\alpha y, \beta z$ and γt can be expressed as

$$\cosh \alpha y = \cos y,\ \sinh \alpha y = \alpha \sin y, \tag{101}$$

$$\cosh \beta y = \cos y,\ \sinh \beta y = \beta \sin y, \tag{102}$$

$$\cosh \gamma y = \cosh y,\ \sinh \gamma y = \gamma \sinh y. \tag{103}$$

The hyperbolic cosine and sine functions of a circular fourcomplex number $x + \alpha y + \beta z + \gamma t$ can then be expressed in terms of elementary functions with the aid of the addition theorems Eqs. (99), (100) and of the expressions in Eqs. (101)-(103).

2.5 Power series of circular fourcomplex variables

A circular fourcomplex series is an infinite sum of the form

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots, \tag{104}$$

where the coefficients a_n are circular fourcomplex numbers. The convergence of the series (104) can be defined in terms of the convergence of its 4 real components. The convergence of a circular fourcomplex series can however be studied using circular fourcomplex variables. The main criterion for absolute convergence remains the comparison theorem, but this requires a number of inequalities which will be discussed further.

The modulus of a circular fourcomplex number $u = x + \alpha y + \beta z + \gamma t$ can be defined as

$$|u| = (x^2 + y^2 + z^2 + t^2)^{1/2}, \tag{105}$$

so that, according to Eq. (17), $d = |u|$. Since $|x| \le |u|, |y| \le |u|, |z| \le |u|, |t| \le |u|$, a property of absolute convergence established via a comparison theorem based on the modulus of the series (104) will ensure the absolute convergence of each real component of that series.

The modulus of the sum $u_1 + u_2$ of the circular fourcomplex numbers u_1, u_2 fulfils the inequality

$$||u_1| - |u_2|| \le |u_1 + u_2| \le |u_1| + |u_2|. \tag{106}$$

For the product the relation is

$$|u_1u_2| \le \sqrt{2}|u_1||u_2|, \tag{107}$$

which replaces the relation of equality extant for regular complex numbers. The equality in Eq. (107) takes place for $x_1 = t_1, y_1 = z_1, x_2 = t_2, y_2 = z_2$ or $x_1 = -t_1, y_1 = -z_1, x_2 = -t_2, y_2 = -z_2$. In Eq. (84), this corresponds to $\psi_1 = 0, \psi_2 = 0$ or $\psi_1 = \pi/2, \psi_2 = \pi/2$. The modulus of a product, which has the property that $0 \le |u_1u_2|$, becomes equal to zero for $x_1 = t_1, y_1 = z_1, x_2 = -t_2, y_2 = -z_2$ or $x_1 = -t_1, y_1 = -z_1, x_2 = t_2, y_2 = z_2$, as discussed after Eq. (16). In Eq. (84), the latter situation corresponds to $\psi_1 = 0, \psi_2 = \pi/2$ or $\psi_1 = 0, \psi_2 = \pi/2$.

It can be shown that

$$x^2 + y^2 + z^2 + t^2 \le |u^2| \le \sqrt{2}(x^2 + y^2 + z^2 + t^2). \tag{108}$$

The left relation in Eq. (108) becomes an equality, $x^2 + y^2 + z^2 + t^2 = |u^2|$, for $xt + yz = 0$. This condition corresponds to $\psi_1 = \psi_2 = \pi/4$ in Eq. (84). The inequality in Eq. (107) implies that

$$|u^l| \le 2^{(l-1)/2}|u|^l. \tag{109}$$

From Eqs. (107) and (109) it results that

$$|au^l| \le 2^{l/2}|a||u|^l. \tag{110}$$

A power series of the circular fourcomplex variable u is a series of the form

$$a_0 + a_1u + a_2u^2 + \cdots + a_lu^l + \cdots. \tag{111}$$

Since

$$\left|\sum_{l=0}^{\infty} a_lu^l\right| \le \sum_{l=0}^{\infty} 2^{l/2}|a_l||u|^l, \tag{112}$$

a sufficient condition for the absolute convergence of this series is that

$$\lim_{l\to\infty} \frac{\sqrt{2}|a_{l+1}||u|}{|a_l|} < 1. \tag{113}$$

Thus the series is absolutely convergent for

$$|u| < c, \tag{114}$$

where

$$c=\lim_{l\to\infty}\frac{|a_l|}{\sqrt{2}|a_{l+1}|}. \tag{115}$$

The convergence of the series (111) can be also studied with the aid of the transformation

$$x+\alpha y+\beta z+\gamma t=\sqrt{2}(e_1\xi+\tilde{e}_1\upsilon+e_2\tau+\tilde{e}_2\zeta), \tag{116}$$

where ξ,υ,τ,ζ have been defined in Eq. (18), and

$$e_1=\frac{1+\gamma}{2},\ \tilde{e}_1=\frac{\alpha+\beta}{2},\ e_2=\frac{1-\gamma}{2},\ \tilde{e}_2=\frac{\alpha-\beta}{2}. \tag{117}$$

The ensemble $e_1,\tilde{e}_1,e_2,\tilde{e}_2$ will be called the canonical circular fourcomplex base, and Eq. (116) gives the canonical form of the circular fourcomplex number. It can be checked that

$$\begin{gathered}e_1^2=e_1,\ \tilde{e}_1^2=-e_1,\ e_1\tilde{e}_1=\tilde{e}_1,\ e_2^2=e_2,\ \tilde{e}_2^2=-e_2,\ e_2\tilde{e}_2=\tilde{e}_2,\\ e_1e_2=0,\ \tilde{e}_1\tilde{e}_2=0,\ e_1\tilde{e}_2=0,\ e_2\tilde{e}_1=0.\end{gathered} \tag{118}$$

The moduli of the bases in Eq. (117) are

$$|e_1|=\frac{1}{\sqrt{2}},\ |\tilde{e}_1|=\frac{1}{\sqrt{2}},\ |e_2|=\frac{1}{\sqrt{2}},\ |\tilde{e}_2|=\frac{1}{\sqrt{2}}, \tag{119}$$

and it can be checked that

$$|x+\alpha y+\beta z+\gamma t|^2=\xi^2+\upsilon^2+\tau^2+\zeta^2. \tag{120}$$

If $u=u'u''$, the components ξ,υ,τ,ζ are related, according to Eqs. (34)-(37) by

$$\xi=\sqrt{2}(\xi'\xi''-\upsilon'\upsilon''),\upsilon=\sqrt{2}(\xi'\upsilon''+\upsilon'\xi''),\tau=\sqrt{2}(\tau'\tau''-\zeta'\zeta''),\zeta=\sqrt{2}(\tau'\zeta''+\zeta'\tau''), \tag{121}$$

which show that, upon multiplication, the components ξ,υ and τ,ζ obey, up to a normalization constant, the same rules as the real and imaginary components of usual, two-dimensional complex numbers.

If the coefficients in Eq. (111) are

$$a_l=a_{l0}+\alpha a_{l1}+\beta a_{l2}+\gamma a_{l3}, \tag{122}$$

and

$$A_{l1}=a_{l0}+a_{l3},\ \tilde{A}_{l1}=a_{l1}+a_{l2},\ A_{l2}=a_{l0}-a_{l3},\ \tilde{A}_{l2}=a_{l1}-a_{l2}, \tag{123}$$

the series (111) can be written as

$$\sum_{l=0}^{\infty} 2^{l/2}\left[(e_1 A_{l1} + \tilde{e}_1 \tilde{A}_{l1})(e_1\xi + \tilde{e}_1 \upsilon)^l + (e_2 A_{l2} + \tilde{e}_2 \tilde{A}_{l2})(e_2\tau + \tilde{e}_2\zeta)^l\right]. \tag{124}$$

Thus, the series in Eqs. (111) and (124) are absolutely convergent for

$$\rho_+ < c_1,\ \rho_- < c_2, \tag{125}$$

where

$$c_1 = \lim_{l\to\infty} \frac{\left[A_{l1}^2 + \tilde{A}_{l1}^2\right]^{1/2}}{\sqrt{2}\left[A_{l+1,1}^2 + \tilde{A}_{l+1,1}^2\right]^{1/2}},\ \ c_2 = \lim_{l\to\infty} \frac{\left[A_{l2}^2 + \tilde{A}_{l2}^2\right]^{1/2}}{\sqrt{2}\left[A_{l+1,2}^2 + \tilde{A}_{l+1,2}^2\right]^{1/2}}. \tag{126}$$

It can be shown that $c = (1/\sqrt{2})\min(c_1, c_2)$, where min designates the smallest of the numbers c_1, c_2. Using the expression of $|u|$ in Eq. (120), it can be seen that the spherical region of convergence defined in Eqs. (114), (115) is included in the cylindrical region of convergence defined in Eqs. (125) and (126).

2.6 Analytic functions of circular fourcomplex variables

The derivative of a function $f(u)$ of the fourcomplex variables u is defined as a function $f'(u)$ having the property that

$$|f(u) - f(u_0) - f'(u_0)(u - u_0)| \to 0 \ \text{as}\ |u - u_0| \to 0. \tag{127}$$

If the difference $u - u_0$ is not parallel to one of the nodal hypersurfaces, the definition in Eq. (127) can also be written as

$$f'(u_0) = \lim_{u\to u_0} \frac{f(u) - f(u_0)}{u - u_0}. \tag{128}$$

The derivative of the function $f(u) = u^m$, with m an integer, is $f'(u) = mu^{m-1}$, as can be seen by developing $u^m = [u_0 + (u - u_0)]^m$ as

$$u^m = \sum_{p=0}^{m} \frac{m!}{p!(m-p)!} u_0^{m-p}(u - u_0)^p, \tag{129}$$

and using the definition (127).

If the function $f'(u)$ defined in Eq. (127) is independent of the direction in space along which u is approaching u_0, the function $f(u)$ is said to be analytic, analogously to the case

of functions of regular complex variables. [7] The function u^m, with m an integer, of the fourcomplex variable u is analytic, because the difference $u^m - u_0^m$ is always proportional to $u - u_0$, as can be seen from Eq. (129). Then series of integer powers of u will also be analytic functions of the fourcomplex variable u, and this result holds in fact for any commutative algebra.

If an analytic function is defined by a series around a certain point, for example $u = 0$, as

$$f(u) = \sum_{k=0}^{\infty} a_k u^k, \tag{130}$$

an expansion of $f(u)$ around a different point u_0,

$$f(u) = \sum_{k=0}^{\infty} c_k (u - u_0)^k, \tag{131}$$

can be obtained by substituting in Eq. (130) the expression of u^k according to Eq. (129). Assuming that the series are absolutely convergent so that the order of the terms can be modified and ordering the terms in the resulting expression according to the increasing powers of $u - u_0$ yields

$$f(u) = \sum_{k,l=0}^{\infty} \frac{(k+l)!}{k!l!} a_{k+l} u_0^l (u - u_0)^k. \tag{132}$$

Since the derivative of order k at $u = u_0$ of the function $f(u)$, Eq. (130), is

$$f^{(k)}(u_0) = \sum_{l=0}^{\infty} \frac{(k+l)!}{l!} a_{k+l} u_0^l, \tag{133}$$

the expansion of $f(u)$ around $u = u_0$, Eq. (132), becomes

$$f(u) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(u_0)(u - u_0)^k, \tag{134}$$

which has the same form as the series expansion of 2-dimensional complex functions. The relation (134) shows that the coefficients in the series expansion, Eq. (131), are

$$c_k = \frac{1}{k!} f^{(k)}(u_0). \tag{135}$$

The rules for obtaining the derivatives and the integrals of the basic functions can be obtained from the series of definitions and, as long as these series expansions have the same form as the corresponding series for the 2-dimensional complex functions, the rules of derivation and integration remain unchanged.

If the fourcomplex function $f(u)$ of the fourcomplex variable u can be expressed in terms of the real functions $P(x,y,z,t), Q(x,y,z,t), R(x,y,z,t), S(x,y,z,t)$ of real variables x,y,z,t as

$$f(u) = P(x,y,z,t) + \alpha Q(x,y,z,t) + \beta R(x,y,z,t) + \gamma S(x,y,z,t), \tag{136}$$

then relations of equality exist between partial derivatives of the functions P,Q,R,S. These relations can be obtained by writing the derivative of the function f as

$$\begin{aligned}\lim_{u\to u_0} \frac{1}{\Delta x + \alpha\Delta y + \beta\Delta z + \gamma\Delta t} &\left[\frac{\partial P}{\partial x}\Delta x + \frac{\partial P}{\partial y}\Delta y + \frac{\partial P}{\partial z}\Delta z + \frac{\partial P}{\partial t}\Delta t\right. \\ &+\alpha\left(\frac{\partial Q}{\partial x}\Delta x + \frac{\partial Q}{\partial y}\Delta y + \frac{\partial Q}{\partial z}\Delta z + \frac{\partial Q}{\partial t}\Delta t\right) + \beta\left(\frac{\partial R}{\partial x}\Delta x + \frac{\partial R}{\partial y}\Delta y + \frac{\partial R}{\partial z}\Delta z + \frac{\partial R}{\partial t}\Delta t\right) \\ &\left.+\gamma\left(\frac{\partial S}{\partial x}\Delta x + \frac{\partial S}{\partial y}\Delta y + \frac{\partial S}{\partial z}\Delta z + \frac{\partial S}{\partial t}\Delta t\right)\right],\end{aligned} \tag{137}$$

where the difference appearing in Eq. (128) is $u - u_0 = \Delta x + \alpha\Delta y + \beta\Delta z + \gamma\Delta t$. These relations have the same form for all systems of hypercomplex numbers discussed in this work.

For the present system of hypercomplex numbers, the relations between the partial derivatives of the functions P,Q,R,S are obtained by setting succesively in Eq. (137) $\Delta x \to 0, \Delta y = \Delta z = \Delta t = 0$; then $\Delta y \to 0, \Delta x = \Delta z = \Delta t = 0$; then $\Delta z \to 0, \Delta x = \Delta y = \Delta t = 0$; and finally $\Delta t \to 0, \Delta x = \Delta y = \Delta z = 0$. The relations are

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} = \frac{\partial R}{\partial z} = \frac{\partial S}{\partial t}, \tag{138}$$

$$\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y} = -\frac{\partial S}{\partial z} = \frac{\partial R}{\partial t}, \tag{139}$$

$$\frac{\partial R}{\partial x} = -\frac{\partial S}{\partial y} = -\frac{\partial P}{\partial z} = \frac{\partial Q}{\partial t}, \tag{140}$$

$$\frac{\partial S}{\partial x} = \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} = \frac{\partial P}{\partial t}. \tag{141}$$

The relations (138)-(141) are analogous to the Riemann relations for the real and imaginary components of a complex function. It can be shown from Eqs. (138)-(141) that the component P is a solution of the equations

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0,\ \ \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} = 0,\ \ \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial t^2} = 0,\ \ \frac{\partial^2 P}{\partial z^2} + \frac{\partial^2 P}{\partial t^2} = 0, \tag{142}$$

$$\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial t^2} = 0,\ \ \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} = 0, \tag{143}$$

and the components Q, R, S are solutions of similar equations.

As can be seen from Eqs. (142)-(143), the components P, Q, R, S of an analytic function of circular fourcomplex variable are harmonic with respect to the pairs of variables $x, y; x, z; y, t$ and z, t, and are solutions of the wave equation with respect to the pairs of variables x, t and y, z. The components P, Q, R, S are also solutions of the mixed-derivative equations

$$\frac{\partial^2 P}{\partial x \partial y} = \frac{\partial^2 P}{\partial z \partial t}, \; \frac{\partial^2 P}{\partial x \partial z} = \frac{\partial^2 P}{\partial y \partial t}, \; \frac{\partial^2 P}{\partial x \partial t} = -\frac{\partial^2 P}{\partial y \partial z}, \tag{144}$$

and the components Q, R, S are solutions of similar equations.

2.7 Integrals of functions of circular fourcomplex variables

The singularities of circular fourcomplex functions arise from terms of the form $1/(u-u_0)^m$, with $m > 0$. Functions containing such terms are singular not only at $u = u_0$, but also at all points of the two-dimensional hyperplanes passing through u_0 and which are parallel to the nodal hyperplanes.

The integral of a circular fourcomplex function between two points A, B along a path situated in a region free of singularities is independent of path, which means that the integral of an analytic function along a loop situated in a region free from singularities is zero,

$$\oint_\Gamma f(u) du = 0, \tag{145}$$

where it is supposed that a surface Σ spanning the closed loop Γ is not intersected by any of the two-dimensional hyperplanes associated with the singularities of the function $f(u)$. Using the expression, Eq. (136), for $f(u)$ and the fact that $du = dx + \alpha dy + \beta dz + \gamma dt$, the explicit form of the integral in Eq. (145) is

$$\begin{aligned}\oint_\Gamma f(u) du = \oint_\Gamma &[(Pdx - Qdy - Rdz + Sdt) + \alpha(Qdx + Pdy + Sdz + Rdt) \\ &+\beta(Rdx + Sdy + Pdz + Qdt) + \gamma(Sdx - Rdy - Qdz + Pdt)].\end{aligned} \tag{146}$$

If the functions P, Q, R, S are regular on a surface Σ spanning the loop Γ, the integral along the loop Γ can be transformed with the aid of the theorem of Stokes in an integral over the surface Σ of terms of the form $\partial P/\partial y + \partial Q/\partial x$, $\partial P/\partial z + \partial R/\partial x$, $\partial P/\partial t - \partial S/\partial x$, $\partial Q/\partial z - \partial R/\partial y$, $\partial Q/\partial t + \partial S/\partial y$, $\partial R/\partial t + \partial S/\partial z$ and of similar terms arising from the α, β and γ components, which are equal to zero by Eqs. (138)-(141), and this proves Eq. (145).

The integral of the function $(u-u_0)^m$ on a closed loop Γ is equal to zero for m a positive or negative integer not equal to -1,

$$\oint_\Gamma (u-u_0)^m du = 0,\ \ m \text{ integer},\ m \neq -1. \tag{147}$$

This is due to the fact that $\int (u-u_0)^m du = (u-u_0)^{m+1}/(m+1)$, and to the fact that the function $(u-u_0)^{m+1}$ is singlevalued for m an integer.

The integral $\oint du/(u-u_0)$ can be calculated using the exponential form (70),

$$u-u_0 = \rho \exp\left(\alpha\frac{\phi+\chi}{2} + \beta\frac{\phi-\chi}{2} + \gamma \ln\tan\psi\right), \tag{148}$$

so that

$$\frac{du}{u-u_0} = \frac{d\rho}{\rho} + \frac{\alpha+\beta}{2}d\phi + \frac{\alpha-\beta}{2}d\chi + \gamma d\ln\tan\psi. \tag{149}$$

Since ρ and $\ln\tan\psi$ are singlevalued variables, it follows that $\oint_\Gamma d\rho/\rho = 0, \oint_\Gamma d\ln\tan\psi = 0$. On the other hand, ϕ and χ are cyclic variables, so that they may give a contribution to the integral around the closed loop Γ. Thus, if C_+ is a circle of radius r parallel to the $\xi O \upsilon$ plane, and the projection of the center of this circle on the $\xi O \upsilon$ plane coincides with the projection of the point u_0 on this plane, the points of the circle C_+ are described according to Eqs. (18)-(22) by the equations

$$\begin{aligned} &\xi = \xi_0 + r\sin\psi\cos\phi,\ \upsilon = \upsilon_0 + r\sin\psi\sin\phi,\ \tau = \tau_0 + r\cos\psi\cos\chi, \\ &\qquad \zeta = \zeta_0 + r\cos\psi\sin\chi, \end{aligned} \tag{150}$$

for constant values of χ and ψ, $\psi \neq 0, \pi/2$, where $u_0 = x_0 + \alpha y_0 + \beta z_0 + \gamma t_0$, and $\xi_0, \upsilon_0, \tau_0, \zeta_0$ are calculated from x_0, y_0, z_0, t_0 according to Eqs. (18). Then

$$\oint_{C_+} \frac{du}{u-u_0} = \pi(\alpha+\beta). \tag{151}$$

If C_- is a circle of radius r parallel to the $\tau O \zeta$ plane, and the projection of the center of this circle on the $\tau O \zeta$ plane coincides with the projection of the point u_0 on this plane, the points of the circle C_- are described by the same Eqs. (150) but for constant values of ϕ and ψ, $\psi \neq 0, \pi/2$ Then

$$\oint_{C_-} \frac{du}{u-u_0} = \pi(\alpha-\beta). \tag{152}$$

The expression of $\oint_\Gamma du/(u-u_0)$ can be written as a single equation with the aid of a functional $\mathrm{int}(M,C)$ defined for a point M and a closed curve C in a two-dimensional plane, such that

$$\mathrm{int}(M,C)=\begin{cases} 1 \ \text{if} \ M \ \text{is an interior point of} \ C, \\ 0 \ \text{if} \ M \ \text{is exterior to} \ C. \end{cases} \tag{153}$$

With this notation the result of the integration along a closed path Γ can be written as

$$\oint_\Gamma \frac{du}{u-u_0}=\pi(\alpha+\beta)\ \mathrm{int}(u_{0\xi\upsilon},\Gamma_{\xi\upsilon})+\pi(\alpha-\beta)\ \mathrm{int}(u_{0\tau\zeta},\Gamma_{\tau\zeta}), \tag{154}$$

where $u_{0\xi\upsilon}, u_{0\tau\zeta}$ and $\Gamma_{\xi\upsilon},\Gamma_{\tau\zeta}$ are respectively the projections of the point u_0 and of the loop Γ on the planes $\xi\upsilon$ and $\tau\zeta$.

If $f(u)$ is an analytic circular fourcomplex function which can be expanded in a series as written in Eq. (131), and the expansion holds on the curve Γ and on a surface spanning Γ, then from Eqs. (147) and (154) it follows that

$$\oint_\Gamma \frac{f(u)du}{u-u_0}=\pi[(\alpha+\beta)\ \mathrm{int}(u_{0\xi\upsilon},\Gamma_{\xi\upsilon})+(\alpha-\beta)\ \mathrm{int}(u_{0\tau\zeta},\Gamma_{\tau\zeta})]\ f(u_0), \tag{155}$$

where $\Gamma_{\xi\upsilon},\Gamma_{\tau\zeta}$ are the projections of the curve Γ on the planes $\xi\upsilon$ and respectively $\tau\zeta$, as shown in Fig. 2. Substituting in the right-hand side of Eq. (155) the expression of $f(u)$ in terms of the real components P,Q,R,S, Eq. (136), yields

$$\begin{aligned}\oint_\Gamma \frac{f(u)du}{u-u_0}=&\ \pi[-(1+\gamma)(Q+R)+(\alpha+\beta)(P+S)]\ \mathrm{int}(u_{0\xi\upsilon},\Gamma_{\xi\upsilon})\\ &+\pi[-(1-\gamma)(Q-R)+(\alpha-\beta)(P-S)]\ \mathrm{int}(u_{0\tau\zeta},\Gamma_{\tau\zeta}),\end{aligned} \tag{156}$$

where P,Q,R,S are the values of the components of f at $u=u_0$.

If $f(u)$ can be expanded as written in Eq. (131) on Γ and on a surface spanning Γ, then from Eqs. (147) and (154) it also results that

$$\oint_\Gamma \frac{f(u)du}{(u-u_0)^{m+1}}=\frac{\pi}{m!}[(\alpha+\beta)\ \mathrm{int}(u_{0\xi\upsilon},\Gamma_{\xi\upsilon})+(\alpha-\beta)\ \mathrm{int}(u_{0\tau\zeta},\Gamma_{\tau\zeta})]\ f^{(m)}(u_0), \tag{157}$$

where it has been used the fact that the derivative $f^{(m)}(u_0)$ of order m of $f(u)$ at $u=u_0$ is related to the expansion coefficient in Eq. (131) according to Eq. (135).

If a function $f(u)$ is expanded in positive and negative powers of $u-u_j$, where u_j are circular fourcomplex constants, j being an index, the integral of f on a closed loop Γ is determined by the terms in the expansion of f which are of the form $a_j/(u-u_j)$,

$$f(u)=\cdots+\sum_j \frac{a_j}{u-u_j}+\cdots \tag{158}$$

Then the integral of f on a closed loop Γ is

$$\oint_\Gamma f(u)du = \pi(\alpha+\beta)\sum_j \mathrm{int}(u_{j\xi\upsilon},\Gamma_{\xi\upsilon})a_j + \pi(\alpha-\beta)\sum_j \mathrm{int}(u_{j\tau\zeta},\Gamma_{\tau\zeta})a_j. \tag{159}$$

2.8 Factorization of circular fourcomplex polynomials

A polynomial of degree m of the circular fourcomplex variable $u = x+\alpha y+\beta z+\gamma t$ has the form

$$P_m(u) = u^m + a_1 u^{m-1} + \cdots + a_{m-1}u + a_m, \tag{160}$$

where the constants are in general circular fourcomplex numbers.

It can be shown that any circular fourcomplex polynomial has a circular fourcomplex root, whence it follows that a polynomial of degree m can be written as a product of m linear factors of the form $u-u_j$, where the circular fourcomplex numbers u_j are the roots of the polynomials, although the factorization may not be unique,

$$P_m(u) = \prod_{j=1}^m (u-u_j). \tag{161}$$

The fact that any circular fourcomplex polynomial has a root can be shown by considering the transformation of a fourdimensional sphere with the center at the origin by the function u^m. The points of the hypersphere of radius d are of the form written in Eq. (80), with d constant and $0\le\phi<2\pi, 0\le\chi<2\pi, 0\le\psi\le\pi/2$. The point u^m is

$$u^m = d^m \exp\left(\alpha m\frac{\phi+\chi}{2} + \beta m\frac{\phi-\chi}{2}\right)[\cos(\psi-\pi/4)+\gamma\sin(\psi-\pi/4)]^m. \tag{162}$$

It can be shown with the aid of Eq. (84) that

$$\left|u\exp\left(\alpha\frac{\phi+\chi}{2}+\beta\frac{\phi-\chi}{2}\right)\right| = |u|, \tag{163}$$

so that

$$\begin{aligned}&\left|[\cos(\psi-\pi/4)+\gamma\sin(\psi-\pi/4)]^m \exp\left(\alpha m\frac{\phi+\chi}{2}+\beta m\frac{\phi-\chi}{2}\right)\right|\\ &\quad = \left|[\cos(\psi-\pi/4)+\gamma\sin(\psi-\pi/4)]^m\right|.\end{aligned} \tag{164}$$

The right-hand side of Eq. (164) is

$$|(\cos\epsilon+\gamma\sin\epsilon)^m|^2 = \sum_{k=0}^m C_{2m}^{2k}\cos^{2m-2k}\epsilon\sin^{2k}\epsilon, \tag{165}$$

where $\epsilon = \psi - \pi/4$, and since $C_{2m}^{2k} \geq C_m^k$, it can be concluded that

$$|(\cos\epsilon + \gamma\sin\epsilon)^m|^2 \geq 1. \tag{166}$$

Then

$$d^m \leq |u^m| \leq 2^{(m-1)/2} d^m, \tag{167}$$

which shows that the image of a four-dimensional sphere via the transformation operated by the function u^m is a finite hypersurface.

If $u' = u^m$, and

$$u' = d'[\cos(\psi' - \pi/4) + \gamma\sin(\psi' - \pi/4)]\exp\left(\alpha\frac{\phi' + \chi'}{2} + \beta\frac{\phi' - \chi'}{2}\right), \tag{168}$$

then

$$\phi' = m\phi,\ \chi' = m\chi,\ \tan\psi' = \tan^m\psi. \tag{169}$$

Since for any values of the angles ϕ', χ', ψ' there is a set of solutions ϕ, χ, ψ of Eqs. (169), and since the image of the hypersphere is a finite hypersurface, it follows that the image of the four-dimensional sphere via the function u^m is also a closed hypersurface. A continuous hypersurface is called closed when any ray issued from the origin intersects that surface at least once in the finite part of the space.

A transformation of the four-dimensional space by the polynomial $P_m(u)$ will be considered further. By this transformation, a hypersphere of radius d having the center at the origin is changed into a certain finite closed surface, as discussed previously. The transformation of the four-dimensional space by the polynomial $P_m(u)$ associates to the point $u = 0$ the point $f(0) = a_m$, and the image of a hypersphere of very large radius d can be represented with good approximation by the image of that hypersphere by the function u^m. The origin of the axes is an inner point of the latter image. If the radius of the hypersphere is now reduced continuously from the initial very large values to zero, the image hypersphere encloses initially the origin, but the image shrinks to a_m when the radius approaches the value zero. Thus, the origin is initially inside the image hypersurface, and it lies outside the image hypersurface when the radius of the hypersphere tends to zero. Then since the image hypersurface is closed, the image surface must intersect at some stage the origin of the axes,

which means that there is a point u_1 such that $f(u_1) = 0$. The factorization in Eq. (161) can then be obtained by iterations.

The roots of the polynomial P_m can be obtained by the following method. If the constants in Eq. (160) are $a_l = a_{l0} + \alpha a_{l1} + \beta a_{l2} + \gamma a_{l3}$, and with the notations of Eq. (123), the polynomial $P_m(u)$ can be written as

$$\begin{aligned}P_m = &\sum_{l=0}^{m} 2^{(m-l)/2}(e_1 A_{l1} + \tilde{e}_1 \tilde{A}_{l1})(e_1\xi + \tilde{e}_1 \upsilon)^{m-l} \\ &+ \sum_{l=0}^{m} 2^{(m-l)/2}(e_2 A_{l2} + \tilde{e}_2 \tilde{A}_{l2})(e_2\tau + \tilde{e}_2 \zeta)^{m-l},\end{aligned} \tag{170}$$

where the constants $A_{lk}, \tilde{A}_{lk}, k = 1, 2$, are real numbers. Each of the polynomials of degree m in $e_1\xi + \tilde{e}_1\upsilon, e_2\tau + \tilde{e}_2\zeta$ in Eq. (170) can always be written as a product of linear factors of the form $e_1(\xi - \xi_p) + \tilde{e}_1(\upsilon - \upsilon_p)$ and respectively $e_2(\tau - \tau_p) + \tilde{e}_2(\zeta - \zeta_p)$, where the constants $\xi_p, \upsilon_p, \tau_p, \zeta_p$ are real,

$$\sum_{l=0}^{m} 2^{(m-l)/2}(e_1 A_{l1} + \tilde{e}_1 \tilde{A}_{l1})(e_1\xi + \tilde{e}_1 \upsilon)^{m-l} = \prod_{p=1}^{m} 2^{m/2} \left\{ e_1(\xi - \xi_p) + \tilde{e}_1(\upsilon - \upsilon_p) \right\}, \tag{171}$$

$$\sum_{l=0}^{m} 2^{(m-l)/2}(e_2 A_{l2} + \tilde{e}_2 \tilde{A}_{l2})(e_2\tau + \tilde{e}_2 \zeta)^{m-l} = \prod_{p=1}^{m} 2^{m/2} \left\{ e_2(\tau - \tau_p) + \tilde{e}_2(\zeta - \zeta_p) \right\}. \tag{172}$$

Due to the relations (118), the polynomial $P_m(u)$ can be written as a product of factors of the form

$$P_m(u) = \prod_{p=1}^{m} 2^{m/2} \left\{ e_1(\xi - \xi_p) + \tilde{e}_1(\upsilon - \upsilon_p) + e_2(\tau - \tau_p) + \tilde{e}_2(\zeta - \zeta_p) \right\}. \tag{173}$$

This relation can be written with the aid of Eq. (116) in the form (161), where

$$u_p = \sqrt{2}(e_1\xi_p + \tilde{e}_1\upsilon_p + e_2\tau_p + \tilde{e}_2\zeta_p). \tag{174}$$

The roots $e_1\xi_p + \tilde{e}_1\upsilon_p$ and $e_2\tau_p + \tilde{e}_2\zeta_p$ defined in Eqs. (171) and respectively (172) may be ordered arbitrarily. This means that Eq. (174) gives sets of m roots $u_1, ..., u_m$ of the polynomial $P_m(u)$, corresponding to the various ways in which the roots $e_1\xi_p + \tilde{e}_1\upsilon_p$ and $e_2\tau_p + \tilde{e}_2\zeta_p$ are ordered according to p for each polynomial. Thus, while the hypercomplex components in Eqs. (171), Eqs. (172) taken separately have unique factorizations, the polynomial $P_m(u)$ can be written in many different ways as a product of linear factors. The result of the circular fourcomplex integration, Eq. (159), is however unique.

If, for example, $P(u)=u^2+1$, the possible factorizations are $P=(u-\tilde{e}_1-\tilde{e}_2)(u+\tilde{e}_1+\tilde{e}_2)$ and $P=(u-\tilde{e}_1+\tilde{e}_2)(u+\tilde{e}_1-\tilde{e}_2)$, which can also be written as $u^2+1=(u-\alpha)(u+\alpha)$ or as $u^2+1=(u-\beta)(u+\beta)$. The result of the circular fourcomplex integration, Eq. (159), is however unique. It can be checked that $(\pm\tilde{e}_1\pm\tilde{e}_2)^2=-e_1-e_2=-1$.

2.9 Representation of circular fourcomplex numbers by irreducible matrices

If T is the unitary matrix,

$$T=\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{175}$$

it can be shown that the matrix TUT^{-1} has the form

$$TUT^{-1}=\begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}, \tag{176}$$

where U is the matrix in Eq. (41) used to represent the circular fourcomplex number u. In Eq. (176), V_1, V_2 are the matrices

$$V_1=\begin{pmatrix} x+t & y+z \\ -y-z & x+t \end{pmatrix},\quad V_2=\begin{pmatrix} x-t & y-z \\ -y+z & x-t \end{pmatrix}. \tag{177}$$

In Eq. (176), the symbols 0 denote the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{178}$$

The relations between the variables $x+t, y+z$,$x-t, y-z$ for the multiplication of circular fourcomplex numbers have been written in Eqs. (34)-(37). The matrix TUT^{-1} provides an irreducible representation [8] of the circular fourcomplex number u in terms of matrices with real coefficients.

3 Hyperbolic Complex Numbers in Four Dimensions

3.1 Operations with hyperbolic fourcomplex numbers

A hyperbolic fourcomplex number is determined by its four components (x,y,z,t). The sum of the hyperbolic fourcomplex numbers (x,y,z,t) and (x',y',z',t') is the hyperbolic fourcomplex number $(x+x',y+y',z+z',t+t')$. The product of the hyperbolic fourcomplex numbers (x,y,z,t) and (x',y',z',t') is defined in this work to be the hyperbolic fourcomplex number $(xx'+yy'+zz'+tt', xy'+yx'+zt'+tz', xz'+zx'+yt'+ty', xt'+tx'+yz'+zy')$.

Hyperbolic fourcomplex numbers and their operations can be represented by writing the hyperbolic fourcomplex number (x,y,z,t) as $u=x+\alpha y+\beta z+\gamma t$, where α,β and γ are bases for which the multiplication rules are

$$\alpha^2=1,\ \beta^2=1,\ \gamma^2=1, \alpha\beta=\beta\alpha=\gamma,\ \alpha\gamma=\gamma\alpha=\beta,\ \beta\gamma=\gamma\beta=\alpha. \tag{179}$$

Two hyperbolic fourcomplex numbers $u=x+\alpha y+\beta z+\gamma t, u'=x'+\alpha y'+\beta z'+\gamma t'$ are equal, $u=u'$, if and only if $x=x', y=y', z=z', t=t'$. If $u=x+\alpha y+\beta z+\gamma t, u'=x'+\alpha y'+\beta z'+\gamma t'$ are hyperbolic fourcomplex numbers, the sum $u+u'$ and the product uu' defined above can be obtained by applying the usual algebraic rules to the sum $(x+\alpha y+\beta z+\gamma t)+(x'+\alpha y'+\beta z'+\gamma t')$ and to the product $(x+\alpha y+\beta z+\gamma t)(x'+\alpha y'+\beta z'+\gamma t')$, and grouping of the resulting terms,

$$u+u'=x+x'+\alpha(y+y')+\beta(z+z')+\gamma(t+t'), \tag{180}$$

$$uu'=xx'+yy'+zz'+tt'+\alpha(xy'+yx'+zt'+tz')+\beta(xz'+zx'+yt'+ty')+\gamma(xt'+tx'+yz'+zy') \tag{181}$$

If u,u',u'' are hyperbolic fourcomplex numbers, the multiplication is associative

$$(uu')u''=u(u'u'') \tag{182}$$

and commutative

$$uu'=u'u, \tag{183}$$

as can be checked through direct calculation. The hyperbolic fourcomplex zero is $0+\alpha\cdot 0+\beta\cdot 0+\gamma\cdot 0$, denoted simply 0, and the hyperbolic fourcomplex unity is $1+\alpha\cdot 0+\beta\cdot 0+\gamma\cdot 0$, denoted simply 1.

The inverse of the hyperbolic fourcomplex number $u=x+\alpha y+\beta z+\gamma t$ is a hyperbolic fourcomplex number $u'=x'+\alpha y'+\beta z'+\gamma t'$ having the property that

$$uu'=1. \tag{184}$$

Written on components, the condition, Eq. (184), is

$$\begin{aligned}
&xx'+yy'+zz'+tt'=1,\\
&yx'+xy'+tz'+zt'=0,\\
&zx'+ty'+xz'+yt'=0,\\
&tx'+zy'+yz'+xt'=0.
\end{aligned} \tag{185}$$

The system (185) has the solution

$$x'=\frac{x(x^2-y^2-z^2-t^2)+2yzt}{\nu}, \tag{186}$$

$$y'=\frac{y(-x^2+y^2-z^2-t^2)+2xzt}{\nu}, \tag{187}$$

$$z'=\frac{z(-x^2-y^2+z^2-t^2)+2xyt}{\nu}, \tag{188}$$

$$t'=\frac{t(-x^2-y^2-z^2+t^2)+2xyz}{\nu}, \tag{189}$$

provided that $\nu\neq 0$, where

$$\nu=x^4+y^4+z^4+t^4-2(x^2y^2+x^2z^2+x^2t^2+y^2z^2+y^2t^2+z^2t^2)+8xyzt. \tag{190}$$

The quantity ν can be written as

$$\nu=ss's''s''', \tag{191}$$

where

$$s=x+y+z+t,\ s'=x-y+z-t,\ s''=x+y-z-t,\ s'''=x-y-z+t. \tag{192}$$

The variables s,s',s'',s''' will be called canonical hyperbolic fourcomplex variables. Then a hyperbolic fourcomplex number $u=x+\alpha y+\beta z+\gamma t$ has an inverse, unless

$$s=0,\ \text{ or }\ s'=0,\ \text{ or }\ s''=0,\ \text{ or }\ s'''=0. \tag{193}$$

For arbitrary values of the variables x, y, z, t, the quantity ν can be positive or negative. If $\nu \geq 0$, the quantity $\mu = \nu^{1/4}$ will be called amplitude of the hyperbolic fourcomplex number $x + \alpha y + \beta z + \gamma t$. The normals of the hyperplanes in Eq. (193) are orthogonal to each other. Because of conditions (193) these hyperplanes will be also called the nodal hyperplanes. It can be shown that if $uu' = 0$ then either $u = 0$, or $u' = 0$, or q, q' belong to pairs of orthogonal hypersurfaces as described further. Thus, divisors of zero exist if one of the hyperbolic fourcomplex numbers u, u' belongs to one of the nodal hyperplanes and the other hyperbolic fourcomplex number belongs to the straight line through the origin which is normal to that hyperplane,

$$x + y + z + t = 0, \text{ and } x' = y' = z' = t', \tag{194}$$

or

$$x - y + z - t = 0, \text{ and } x' = -y' = z' = -t', \tag{195}$$

or

$$x + y - z - t = 0, \text{ and } x' = y' = -z' = -t', \tag{196}$$

or

$$x - y - z + t = 0, \text{ and } x' = -y' = -z' = t'. \tag{197}$$

Divisors of zero also exist if the hyperbolic fourcomplex numbers u, u' belong to different members of the pairs of two-dimensional hypersurfaces listed further,

$$x + y = 0, \ z + t = 0 \text{ and } x' - y' = 0, \ z' - t' = 0, \tag{198}$$

or

$$x + z = 0, \ y + t = 0 \text{ and } x' - z' = 0, \ y' - t' = 0, \tag{199}$$

or

$$y + z = 0, \ x + t = 0 \text{ and } y' - z' = 0, \ x' - t' = 0. \tag{200}$$

3.2 Geometric representation of hyperbolic fourcomplex numbers

The hyperbolic fourcomplex number $x+\alpha y+\beta z+\gamma t$ can be represented by the point A of coordinates (x,y,z,t). If O is the origin of the four-dimensional space x,y,z,t, the distance from A to the origin O can be taken as

$$d^2 = x^2+y^2+z^2+t^2. \tag{201}$$

The distance d will be called modulus of the hyperbolic fourcomplex number $x+\alpha y+\beta z+\gamma t$, $d=|u|$.

If $u=x+\alpha y+\beta z+\gamma t, u_1=x_1+\alpha y_1+\beta z_1+\gamma t_1, u_2=x_2+\alpha y_2+\beta z_2+\gamma t_2$, and $u=u_1u_2$, and if

$$s_j=x_j+y_j+z_j+t_j, s_j'=x_j-y_j+z_j-t_j, s_j''=x_j+y_j-z_j-t_j, s_j'''=x_j-y_j-z_j+t_j, j=1,2, \tag{202}$$

it can be shown that

$$s=s_1s_2,\;\; s'=s_1's_2',\;\; s''=s_1''s_2'',\;\; s'''=s_1'''s_2'''. \tag{203}$$

The relations (203) are a consequence of the identities

$$\begin{aligned}&(x_1x_2+y_1y_2+z_1z_2+t_1t_2)+(x_1y_2+y_1x_2+z_1t_2+t_1z_2)\\&\quad+(x_1z_2+z_1x_2+y_1t_2+t_1y_2)+(x_1t_2+t_1x_2+y_1z_2+z_1y_2)\\&\quad=(x_1+y_1+z_1+t_1)(x_2+y_2+z_2+t_2)\end{aligned} \tag{204}$$

$$\begin{aligned}&(x_1x_2+y_1y_2+z_1z_2+t_1t_2)-(x_1y_2+y_1x_2+z_1t_2+t_1z_2)\\&\quad+(x_1z_2+z_1x_2+y_1t_2+t_1y_2)-(x_1t_2+t_1x_2+y_1z_2+z_1y_2)\\&\quad=(x_1-y_1+z_1-t_1)(x_2-y_2+z_2-t_2)\end{aligned} \tag{205}$$

$$\begin{aligned}&(x_1x_2+y_1y_2+z_1z_2+t_1t_2)+(x_1y_2+y_1x_2+z_1t_2+t_1z_2)\\&\quad-(x_1z_2+z_1x_2+y_1t_2+t_1y_2)-(x_1t_2+t_1x_2+y_1z_2+z_1y_2)\\&\quad=(x_1+y_1-z_1-t_1)(x_2+y_2-z_2-t_2)\end{aligned} \tag{206}$$

$$(x_1x_2+y_1y_2+z_1z_2+t_1t_2)-(x_1y_2+y_1x_2+z_1t_2+t_1z_2)$$
$$-(x_1z_2+z_1x_2+y_1t_2+t_1y_2)+(x_1t_2+t_1x_2+y_1z_2+z_1y_2)$$
$$=(x_1-y_1-z_1+t_1)(x_2-y_2-z_2+t_2) \tag{207}$$

A consequence of the relations (203) is that if $u=u_1u_2$, then

$$\nu=\nu_1\nu_2, \tag{208}$$

where

$$\nu_j=s_js_j's_j''s_j''',\ j=1,2. \tag{209}$$

The hyperbolic fourcomplex numbers

$$e=\frac{1+\alpha+\beta+\gamma}{4},\ e'=\frac{1-\alpha+\beta-\gamma}{4},\ e''=\frac{1+\alpha-\beta-\gamma}{4},\ e'''=\frac{1-\alpha-\beta+\gamma}{4} \tag{210}$$

are orthogonal,

$$ee'=0,\ ee''=0,\ ee'''=0,\ e'e''=0,\ e'e'''=0,\ e''e'''=0, \tag{211}$$

and have also the property that

$$e^2=e,\ e'^2=e',\ e''^2=e'',\ e'''^2=e'''. \tag{212}$$

The hyperbolic fourcomplex number $u=x+\alpha y+\beta z+\gamma t$ can be written as

$$x+\alpha y+\beta z+\gamma t=(x+y+z+t)e+(x-y+z-t)e'+(x+y-z-t)e''+(x-y-z+t)e''', \tag{213}$$

or, by using Eq. (192),

$$u=se+s'e'+s''e''+s'''e'''. \tag{214}$$

The ensemble e,e',e'',e''' will be called the canonical hyperbolic fourcomplex base, and Eq. (214) gives the canonical form of the hyperbolic fourcomplex number. Thus, if $u_j=s_je+s_j'e'+s_j''e''+s_j'''e''',\ j=1,2$, and $u=u_1u_2$, then the multiplication of the hyperbolic fourcomplex numbers is expressed by the relations (203). The moduli of the bases e,e',e'',e''' are

$$|e|=\frac{1}{2},\ |e'|=\frac{1}{2},\ |e''|=\frac{1}{2},\ |e'''|=\frac{1}{2}. \tag{215}$$

The distance d, Eq. (201), is given by

$$d^2=\frac{1}{4}\left(s^2+s'^2+s''^2+s'''^2\right). \tag{216}$$

The relation (216) shows that the variables s, s', s'', s''' can be written as

$$s=2d\cos\psi\cos\phi,\ s'=2d\cos\psi\sin\phi,\ s''=2d\sin\psi\cos\chi,\ s=2d\sin\psi\sin\chi, \tag{217}$$

where ϕ is the azimuthal angle in the s, s' plane, $0\leq\phi<2\pi$, χ is the azimuthal angle in the s'', s''' plane, $0\leq\chi<2\pi$, and ψ is the angle between the line OA and the plane ss', $0\leq\psi\leq\pi/2$. The variables x, y, z, t can be expressed in terms of the distance d and the angles ϕ, χ, ψ as

$$\begin{aligned}
x&=\tfrac{d}{2}(\cos\psi\cos\phi+\cos\psi\sin\phi+\sin\psi\cos\chi+\sin\psi\sin\chi),\\
y&=\tfrac{d}{2}(\cos\psi\cos\phi-\cos\psi\sin\phi+\sin\psi\cos\chi-\sin\psi\sin\chi),\\
z&=\tfrac{d}{2}(\cos\psi\cos\phi+\cos\psi\sin\phi-\sin\psi\cos\chi-\sin\psi\sin\chi),\\
t&=\tfrac{d}{2}(\cos\psi\cos\phi-\cos\psi\sin\phi-\sin\psi\cos\chi+\sin\psi\sin\chi).
\end{aligned} \tag{218}$$

If $u=u_1u_2$, and the hypercomplex numbers u_1, u_2 are described by the variables $d_1, \phi_1, \chi_1, \psi_1$ and respectively $d_2, \phi_2, \chi_2, \psi_2$, then from Eq. (217) it results that

$$\begin{gathered}
\tan\phi=\tan\phi_1\tan\phi_2,\ \tan\chi=\tan\chi_1\tan\chi_2,\\
\frac{\tan^2\psi\sin 2\chi}{\sin 2\phi}=\frac{\tan^2\psi_1\sin 2\chi_1}{\sin 2\phi_1}\frac{\tan^2\psi_2\sin 2\chi_2}{\sin 2\phi_2}.
\end{gathered} \tag{219}$$

The relation (208) for the product of hyperbolic fourcomplex numbers can be demonstrated also by using a representation of the multiplication of the hyperbolic fourcomplex numbers by matrices, in which the hyperbolic fourcomplex number $u=x+\alpha y+\beta z+\gamma t$ is represented by the matrix

$$A=\begin{pmatrix} x & y & z & t\\ y & x & t & z\\ z & t & x & y\\ t & z & y & x\end{pmatrix}. \tag{220}$$

The product $u=x+\alpha y+\beta z+\gamma t$ of the hyperbolic fourcomplex numbers $u_1=x_1+\alpha y_1+\beta z_1+\gamma t_1, u_2=x_2+\alpha y_2+\beta z_2+\gamma t_2$, can be represented by the matrix multiplication

$$A=A_1A_2. \tag{221}$$

It can be checked that the determinant $\det(A)$ of the matrix A is

$$\det A = \nu. \tag{222}$$

The identity (208) is then a consequence of the fact the determinant of the product of matrices is equal to the product of the determinants of the factor matrices.

3.3 Exponential form of a hyperbolic fourcomplex number

The exponential function of a fourcomplex variable u and the addition theorem for the exponential function have been written in Eqs. (44) and (45). If $u = x + \alpha y + \beta z + \gamma t$, then $\exp u$ can be calculated as $\exp u = \exp x \cdot \exp(\alpha y) \cdot \exp(\beta z) \cdot \exp(\gamma t)$. According to Eqs. (179),

$$\alpha^{2m} = 1, \alpha^{2m+1} = \alpha, \beta^{2m} = 1, \beta^{2m+1} = \beta, \gamma^{2m} = 1, \gamma^{2m+1} = \gamma, \tag{223}$$

where m is a natural number, so that $\exp(\alpha y)$, $\exp(\beta z)$ and $\exp(\gamma t)$ can be written as

$$\exp(\alpha y) = \cosh y + \alpha \sinh y, \exp(\beta z) = \cosh y + \beta \sinh z, \exp(\gamma t) = \cosh t + \gamma \sinh t. \tag{224}$$

From Eqs. (224) it can be inferred that

$$(\cosh t + \alpha \sinh t)^m = \cosh mt + \alpha \sinh mt,\ (\cosh t + \beta \sinh t)^m = \cosh mt + \beta \sinh mt,$$
$$(\cosh t + \gamma \sinh t)^m = \cosh mt + \gamma \sinh mt. \tag{225}$$

The hyperbolic fourcomplex numbers $u = x + \alpha y + \beta z + \gamma t$ for which $s = x + y + z + t > 0$, $s' = x - y + z - t > 0$, $s'' = x + y - z - t > 0$, $s''' = x - y - z + t > 0$ can be written in the form

$$x + \alpha y + \beta z + \gamma t = e^{x_1 + \alpha y_1 + \beta z_1 + \gamma t_1}. \tag{226}$$

The conditions $s = x + y + z + t > 0$, $s' = x - y + z - t > 0$, $s'' = x + y - z - t > 0$, $s''' = x - y - z + t > 0$ correspond in Eq. (217) to a range of angles $0 < \phi < \pi/2, 0 < \chi < \pi/2, 0 < \psi \leq \pi/2$. The expressions of x_1, y_1, z_1, t_1 as functions of x, y, z, t can be obtained by developing $e^{\alpha y_1}, e^{\beta z_1}$and $e^{\gamma t_1}$ with the aid of Eqs. (224), by multiplying these expressions and separating the hypercomplex components,

$$x = e^{x_1}(\cosh y_1 \cosh z_1 \cosh t_1 + \sinh y_1 \sinh z_1 \sinh t_1), \tag{227}$$

$$y = e^{x_1}(\sinh y_1 \cosh z_1 \cosh t_1 + \cosh y_1 \sinh z_1 \sinh t_1), \tag{228}$$

$$z = e^{x_1}(\cosh y_1 \sinh z_1 \cosh t_1 + \sinh y_1 \cosh z_1 \sinh t_1), \tag{229}$$

$$t = e^{x_1}(\sinh y_1 \sinh z_1 \cosh t_1 + \cosh y_1 \cosh z_1 \sinh t_1), \tag{230}$$

It can be shown from Eqs. (227)-(230) that

$$x_1 = \frac{1}{4}\ln(ss's''s'''),\ y_1 = \frac{1}{4}\ln\frac{ss''}{s's'''},\ z_1 = \frac{1}{4}\ln\frac{ss'}{s''s'''},\ t_1 = \frac{1}{4}\ln\frac{ss'''}{s's''}. \tag{231}$$

The exponential form of the hyperbolic fourcomplex number u can be written as

$$u = \mu \exp\left(\frac{1}{4}\alpha\ln\frac{ss''}{s's'''} + \frac{1}{4}\beta\ln\frac{ss'}{s''s'''} + \frac{1}{4}\gamma\ln\frac{ss'''}{s's''}\right), \tag{232}$$

where

$$\mu = (ss's''s''')^{1/4}. \tag{233}$$

The exponential form of the hyperbolic fourcomplex number u can be written with the aid of the relations (217) as

$$u = \mu \exp\left(\frac{1}{4}\alpha\ln\frac{1}{\tan\phi\tan\chi} + \frac{1}{4}\beta\ln\frac{\sin 2\phi}{\tan^2\psi\sin 2\chi} + \frac{1}{4}\gamma\ln\frac{\tan\chi}{\tan\phi}\right). \tag{234}$$

The amplitude μ can be expressed in terms of the distance d with the aid of Eqs. (217) as

$$\mu = d\sin^{1/2}2\psi\sin^{1/4}2\phi\sin^{1/4}2\chi. \tag{235}$$

The hypercomplex number can be written as

$$u = d\sin^{1/2}2\psi\sin^{1/4}2\phi\sin^{1/4}2\chi\exp\left(\frac{1}{4}\alpha\ln\frac{1}{\tan\phi\tan\chi} + \frac{1}{4}\beta\ln\frac{\sin 2\phi}{\tan^2\psi\sin 2\chi} + \frac{1}{4}\gamma\ln\frac{\tan\chi}{\tan\phi}\right), \tag{236}$$

which is the trigonometric form of the hypercomplex number u.

3.4 Elementary functions of a hyperbolic fourcomplex variable

The logarithm u_1 of the hyperbolic fourcomplex number u, $u_1 = \ln u$, can be defined for $s > 0, s' > 0, s'' > 0, s''' > 0$ as the solution of the equation

$$u = e^{u_1}, \tag{237}$$

for u_1 as a function of u. From Eq. (232) it results that

$$\ln u = \frac{1}{4}\ln\mu + +\frac{1}{4}\alpha\ln\frac{ss''}{s's'''} + \frac{1}{4}\beta\ln\frac{ss'}{s''s'''} + \frac{1}{4}\gamma\ln\frac{ss'''}{s's''}. \tag{238}$$

Using the expression in Eq. (234), the logarithm can be written as

$$\ln u = \frac{1}{4}\ln\mu + \frac{1}{4}\alpha\ln\frac{1}{\tan\phi\tan\chi} + \frac{1}{4}\beta\ln\frac{\sin 2\phi}{\tan^2\psi\sin 2\chi} + \frac{1}{4}\gamma\ln\frac{\tan\chi}{\tan\phi}. \tag{239}$$

It can be inferred from Eqs. (238) and (203) that

$$\ln(u_1u_2) = \ln u_1 + \ln u_2. \tag{240}$$

The explicit form of Eq. (238) is

$$\begin{aligned}\ln(x+\alpha y+\beta z+\gamma t) &= \frac{1}{4}(1+\alpha+\beta+\gamma)\ln(x+y+z+t)\\ &+\frac{1}{4}(1-\alpha+\beta-\gamma)\ln(x-y+z-t)+\frac{1}{4}(1+\alpha-\beta-\gamma)\ln(x+y-z-t)\\ &+\frac{1}{4}(1-\alpha-\beta+\gamma)\ln(x-y-z+t).\end{aligned} \tag{241}$$

The relation (241) can be written with the aid of Eq. (210) as

$$\ln u = e\ln s + e'\ln s' + e''\ln s'' + e'''\ln s'''. \tag{242}$$

The power function u^n can be defined for $s>0, s'>0, s''>0, s'''>0$ and real values of n as

$$u^n = e^{n\ln u}. \tag{243}$$

It can be inferred from Eqs. (243) and (240) that

$$(u_1u_2)^n = u_1^n\,u_2^n. \tag{244}$$

Using the expression (241) for $\ln u$ and the relations (211) and (212) it can be shown that

$$\begin{aligned}(x+\alpha y+\beta z+\gamma t)^n &= \frac{1}{4}(1+\alpha+\beta+\gamma)(x+y+z+t)^n + \frac{1}{4}(1-\alpha+\beta-\gamma)(x-y+z-t)^n\\ &+\frac{1}{4}(1+\alpha-\beta-\gamma)(x+y-z-t)^n + \frac{1}{4}(1-\alpha-\beta+\gamma)(x-y-z+t)^n.\end{aligned} \tag{245}$$

For integer n, the relation (245) is valid for any x,y,z,t. The relation (245) for $n=-1$ is

$$\frac{1}{x+\alpha y+\beta z+\gamma t} = \frac{1}{4}\left(\frac{1+\alpha+\beta+\gamma}{x+y+z+t}+\frac{1-\alpha+\beta-\gamma}{x-y+z-t}+\frac{1+\alpha-\beta-\gamma}{x+y-z-t}+\frac{1-\alpha-\beta+\gamma}{x-y-z+t}\right). \tag{246}$$

The trigonometric functions of the fourcomplex variable u and the addition theorems for these functions have been written in Eqs. (90)-(93). The cosine and sine functions of the hypercomplex variables $\alpha y, \beta z$ and γt can be expressed as

$$\cos\alpha y = \cos y,\ \sin\alpha y = \alpha\sin y, \tag{247}$$

$$\cos\beta y = \cos y,\ \sin\beta y = \beta\sin y, \tag{248}$$

$$\cos\gamma y = \cos y,\ \sin\gamma y = \gamma\sin y. \tag{249}$$

The cosine and sine functions of a hyperbolic fourcomplex number $x+\alpha y+\beta z+\gamma t$ can then be expressed in terms of elementary functions with the aid of the addition theorems Eqs. (92), (93) and of the expressions in Eqs. (247)-(249).

The hyperbolic functions of the fourcomplex variable u and the addition theorems for these functions have been written in Eqs. (97)-(100). The hyperbolic cosine and sine functions of the hypercomplex variables $\alpha y, \beta z$ and γt can be expressed as

$$\cosh\alpha y = \cosh y,\ \sinh\alpha y = \alpha\sinh y, \tag{250}$$

$$\cosh\beta y = \cosh y,\ \sinh\beta y = \beta\sinh y, \tag{251}$$

$$\cosh\gamma y = \cosh y,\ \sinh\gamma y = \gamma\sinh y. \tag{252}$$

The hyperbolic cosine and sine functions of a hyperbolic fourcomplex number $x+\alpha y+\beta z+\gamma t$ can then be expressed in terms of elementary functions with the aid of the addition theorems Eqs. (99), (100) and of the expressions in Eqs. (250)-(252).

3.5 Power series of hyperbolic fourcomplex variables

A hyperbolic fourcomplex series is an infinite sum of the form

$$a_0 + a_1 + a_2 + \cdots + a_l + \cdots, \tag{253}$$

where the coefficients a_l are hyperbolic fourcomplex numbers. The convergence of the series (253) can be defined in terms of the convergence of its 4 real components. The convergence of a hyperbolic fourcomplex series can however be studied using hyperbolic fourcomplex

variables. The main criterion for absolute convergence remains the comparison theorem, but this requires a number of inequalities which will be discussed further.

The modulus of a hyperbolic fourcomplex number $u = x + \alpha y + \beta z + \gamma t$ can be defined as

$$|u| = (x^2 + y^2 + z^2 + t^2)^{1/2}, \tag{254}$$

so that according to Eq. (201) $d = |u|$. Since $|x| \le |u|, |y| \le |u|, |z| \le |u|, |t| \le |u|$, a property of absolute convergence established via a comparison theorem based on the modulus of the series (253) will ensure the absolute convergence of each real component of that series.

The modulus of the sum $u_1 + u_2$ of the hyperbolic fourcomplex numbers u_1, u_2 fulfils the inequality

$$||u_1| - |u_2|| \le |u_1 + u_2| \le |u_1| + |u_2|. \tag{255}$$

For the product the relation is

$$|u_1 u_2| \le 2|u_1||u_2|, \tag{256}$$

which replaces the relation of equality extant for regular complex numbers. The equality in Eq. (256) takes place for $x_1^2 = y_1^2 = z_1^2 = t_1^2$ and $x_2/x_1 = y_2/y_1 = z_2/z_1 = t_2/t_1$. In particular

$$|u^2| \le 2(x^2 + y^2 + z^2 + t^2). \tag{257}$$

The inequality in Eq. (256) implies that

$$|u^l| \le 2^{l-1}|u|^l. \tag{258}$$

From Eqs. (256) and (258) it results that

$$|au^l| \le 2^l |a||u|^l. \tag{259}$$

A power series of the hyperbolic fourcomplex variable u is a series of the form

$$a_0 + a_1 u + a_2 u^2 + \cdots + a_l u^l + \cdots. \tag{260}$$

Since

$$\left|\sum_{l=0}^{\infty} a_l u^l\right| \le \sum_{l=0}^{\infty} 2^l |a_l||u|^l, \tag{261}$$

a sufficient condition for the absolute convergence of this series is that

$$\lim_{l\to\infty} \frac{2|a_{l+1}||u|}{|a_l|} < 1. \tag{262}$$

Thus the series is absolutely convergent for

$$|u| < c_0, \tag{263}$$

where

$$c_0 = \lim_{l\to\infty} \frac{|a_l|}{2|a_{l+1}|}. \tag{264}$$

The convergence of the series (260) can be also studied with the aid of the formula (245) which, for integer values of l, is valid for any x, y, z, t. If $a_l = a_{lx} + \alpha a_{ly} + \beta a_{lz} + \gamma a_{lt}$, and

$$A_l = a_{lx} + a_{ly} + a_{lz} + a_{lt}, \tag{265}$$

$$A'_l = a_{lx} - a_{ly} + a_{lz} - a_{lt}, \tag{266}$$

$$A''_l = a_{lx} + a_{ly} - a_{lz} - a_{lt}, \tag{267}$$

$$A'''_l = a_{lx} - a_{ly} - a_{lz} + a_{lt}, \tag{268}$$

it can be shown with the aid of relations (211) and (212) that

$$a_l e = A_l e,\ a_l e' = A'_l e',\ a_l e'' = A''_l e'',\ a_l e''' = A'''_l e''', \tag{269}$$

so that the expression of the series (260) becomes

$$\sum_{l=0}^{\infty} \left(A_l s^l e + A'_l s'^l e' + A''_l s''^l e'' + A'''_l s'''^l e''' \right), \tag{270}$$

where the quantities s, s', s'', s''' have been defined in Eq. (192). The sufficient conditions for the absolute convergence of the series in Eq. (270) are that

$$\lim_{l\to\infty} \frac{|A_{l+1}||s|}{|A_l|} < 1, \lim_{l\to\infty} \frac{|A'_{l+1}||s'|}{|A'_l|} < 1, \lim_{l\to\infty} \frac{|A''_{l+1}||s''|}{|A''_l|} < 1, \lim_{l\to\infty} \frac{|A'''_{l+1}||s'''|}{|A'''_l|} < 1, \tag{271}$$

Thus the series in Eq. (270) is absolutely convergent for

$$|x + y + z + t| < c,\ |x - y + z - t| < c',\ |x + y - z - t| < c'',\ |x - y - z + t| < c''', \tag{272}$$

where

$$c = \lim_{l\to\infty} \frac{|A_l|}{|A_{l+1}|},\ c' = \lim_{l\to\infty} \frac{|A'_l|}{|A'_{l+1}|},\ c'' = \lim_{l\to\infty} \frac{|A''_l|}{|A''_{l+1}|},\ c''' = \lim_{l\to\infty} \frac{|A'''_l|}{|A'''_{l+1}|}. \tag{273}$$

The relations (272) show that the region of convergence of the series (270) is a four-dimensional parallelepiped. It can be shown that $c_0 = (1/2)\min(c, c', c'', c''')$, where min designates the smallest of the numbers c, c', c'', c'''. Using Eq. (216), it can be seen that the circular region of convergence defined in Eqs. (263), (264) is included in the parallelogram defined in Eqs. (272) and (273).

3.6 Analytic functions of hyperbolic fourcomplex variables

The fourcomplex function $f(u)$ of the fourcomplex variable u has been expressed in Eq. (136) in terms of the real functions $P(x,y,z,t), Q(x,y,z,t), R(x,y,z,t), S(x,y,z,t)$ of real variables x, y, z, t. The relations between the partial derivatives of the functions P, Q, R, S are obtained by setting succesively in Eq. (137) $\Delta x \to 0, \Delta y = \Delta z = \Delta t = 0$; then $\Delta y \to 0, \Delta x = \Delta z = \Delta t = 0$; then $\Delta z \to 0, \Delta x = \Delta y = \Delta t = 0$; and finally $\Delta t \to 0, \Delta x = \Delta y = \Delta z = 0$. The relations are

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} = \frac{\partial R}{\partial z} = \frac{\partial S}{\partial t}, \tag{274}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial S}{\partial z} = \frac{\partial R}{\partial t}, \tag{275}$$

$$\frac{\partial R}{\partial x} = \frac{\partial S}{\partial y} = \frac{\partial P}{\partial z} = \frac{\partial Q}{\partial t}, \tag{276}$$

$$\frac{\partial S}{\partial x} = \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} = \frac{\partial P}{\partial t}. \tag{277}$$

The relations (274)-(277) are analogous to the Riemann relations for the real and imaginary components of a complex function. It can be shown from Eqs. (274)-(277) that the component P is a solution of the equations

$$\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial y^2} = 0,\ \ \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial z^2} = 0,\ \ \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial t^2} = 0,\ \ \frac{\partial^2 P}{\partial z^2} - \frac{\partial^2 P}{\partial t^2} = 0, \tag{278}$$

$$\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial t^2} = 0,\ \ \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} = 0, \tag{279}$$

and the components Q, R, S are solutions of similar equations. As can be seen from Eqs. (278)-(279), the components P, Q, R, S of an analytic function of hyperbolic fourcomplex

variable are solutions of the wave equation with respect to pairs of the variables x, y, z, t. The component P is also a solution of the mixed-derivative equations

$$\frac{\partial^2 P}{\partial x \partial y} = \frac{\partial^2 P}{\partial z \partial t}, \ \frac{\partial^2 P}{\partial x \partial z} = \frac{\partial^2 P}{\partial y \partial t}, \ \frac{\partial^2 P}{\partial x \partial t} = \frac{\partial^2 P}{\partial y \partial z}, \tag{280}$$

and the components Q, R, S are solutions of similar equations.

3.7 Integrals of functions of hyperbolic fourcomplex variables

The singularities of hyperbolic fourcomplex functions arise from terms of the form $1/(u - u_0)^m$, with $m > 0$. Functions containing such terms are singular not only at $u = u_0$, but also at all points of the two-dimensional hyperplanes passing through u_0 and which are parallel to the nodal hyperplanes.

The integral of a hyperbolic fourcomplex function between two points A, B along a path situated in a region free of singularities is independent of path, which means that the integral of an analytic function along a loop situated in a region free from singularities is zero,

$$\oint_\Gamma f(u) du = 0, \tag{281}$$

where it is supposed that a surface Σ spanning the closed loop Γ is not intersected by any of the two-dimensional hyperplanes associated with the singularities of the function $f(u)$. Using the expression, Eq. (136), for $f(u)$ and the fact that $du = dx + \alpha dy + \beta dz + \gamma dt$, the explicit form of the integral in Eq. (281) is

$$\begin{aligned}\oint_\Gamma f(u) du = \oint_\Gamma [&(Pdx + Qdy + Rdz + Sdt) + \alpha(Qdx + Pdy + Sdz + Rdt) \\ &+\beta(Rdx + Sdy + Pdz + Qdt) + \gamma(Sdx + Rdy + Qdz + Pdt)].\end{aligned} \tag{282}$$

If the functions P, Q, R, S are regular on a surface Σ spanning the loop Γ, the integral along the loop Γ can be transformed with the aid of the theorem of Stokes in an integral over the surface Σ of terms of the form $\partial P/\partial y - \partial Q/\partial x, \ \partial P/\partial z - \partial R/\partial x, \ \partial P/\partial t - \partial S/\partial x, \ \partial Q/\partial z - \partial R/\partial y, \ \partial Q/\partial t - \partial S/\partial y, \ \partial R/\partial t - \partial S/\partial z$ and of similar terms arising from the α, β and γ components, which are equal to zero by Eqs. (274)-(277), and this proves Eq. (281).

The exponential form of the hyperbolic fourcomplex numbers, Eq. (234), contains no cyclic variable, and therefore the concept of residue is not applicable to the hyperbolic fourcomplex numbers defined in Eqs. (179).

3.8 Factorization of hyperbolic fourcomplex polynomials

A polynomial of degree m of the hyperbolic fourcomplex variable $u = x + \alpha y + \beta z + \gamma t$ has the form

$$P_m(u) = u^m + a_1 u^{m-1} + \cdots + a_{m-1} u + a_m, \tag{283}$$

where the constants are in general hyperbolic fourcomplex numbers. If $a_m = a_{mx} + \alpha a_{my} + \beta a_{mz} + \gamma a_{mt}$, and with the notations of Eqs. (192) and (265)-(268) applied for $l = 0, 1, \cdots, m$, the polynomial $P_m(u)$ can be written as

$$\begin{aligned} P_m = & \left[s^m + A_1 s^{m-1} + \cdots + A_{m-1} s + A_m\right] e + \left[s'^m + A'_1 s'^{m-1} + \cdots + A'_{m-1} s' + A'_m\right] e' \\ & + \left[s''^m + A''_1 s''^{m-1} + \cdots + A''_{m-1} s'' + A''_m\right] e'' + \left[s'''^m + A'''_1 s'''^{m-1} + \cdots + A'''_{m-1} s''' + A'''_m\right] e'''. \end{aligned} \tag{284}$$

Each of the polynomials of degree m with real coefficients in Eq. (284) can be written as a product of linear or quadratic factors with real coefficients, or as a product of linear factors which, if imaginary, appear always in complex conjugate pairs. Using the latter form for the simplicity of notations, the polynomial P_m can be written as

$$P_m = \prod_{l=1}^{m} (s - s_l) e + \prod_{l=1}^{m} (s' - s'_l) e' + \prod_{l=1}^{m} (s'' - s''_l) e'' + \prod_{l=1}^{m} (s''' - s'''_l) e''', \tag{285}$$

where the quantities s_l appear always in complex conjugate pairs, and the same is true for the quantities s'_l, for the quantitites s''_l, and for the quantities s'''_l. Due to the properties in Eqs. (211) and (212), the polynomial $P_m(u)$ can be written as a product of factors of the form

$$P_m(u) = \prod_{l=1}^{m} \left[(s - s_l) e + (s' - s'_l) e' + (s'' - s''_l) e'' + (s''' - s'''_l) e'''\right]. \tag{286}$$

These relations can be written with the aid of Eq. (213) as

$$P_m(u) = \prod_{p=1}^{m} (u - u_p), \tag{287}$$

where

$$u_p = s_p e + s'_p e' + s''_p e'' + s'''_p e'''. \tag{288}$$

The roots s_p, s'_p, s''_p, s'''_p of the corresponding polynomials in Eq. (285) may be ordered arbitrarily. This means that Eq. (288) gives sets of m roots $u_1, ..., u_m$ of the polynomial $P_m(u)$,

corresponding to the various ways in which the roots s_p, s'_p, s''_p, s'''_p are ordered according to p in each group. Thus, while the hypercomplex components in Eq. (284) taken separately have unique factorizations, the polynomial $P_m(u)$ can be written in many different ways as a product of linear factors.

If $P(u) = u^2 - 1$, the factorization in Eq. (287) is $u^2 - 1 = (u - u_1)(u - u_2)$, where $u_1 = \pm e \pm e' \pm e'' \pm e'''$, $u_2 = -u_1$, so that there are 8 distinct factorizations of $u^2 - 1$,

$$\begin{aligned}
u^2 - 1 &= (u - e - e' - e'' - e''')\,(u + e + e' + e'' + e'''),\\
u^2 - 1 &= (u - e - e' - e'' + e''')\,(u + e + e' + e'' - e'''),\\
u^2 - 1 &= (u - e - e' + e'' - e''')\,(u + e + e' - e'' + e'''),\\
u^2 - 1 &= (u - e + e' - e'' - e''')\,(u + e - e' + e'' + e'''),\\
u^2 - 1 &= (u - e - e' + e'' + e''')\,(u + e + e' - e'' - e'''),\\
u^2 - 1 &= (u - e + e' - e'' + e''')\,(u + e - e' + e'' - e'''),\\
u^2 - 1 &= (u - e + e' + e'' - e''')\,(u + e - e' - e'' + e'''),\\
u^2 - 1 &= (u - e + e' + e'' + e''')\,(u + e - e' - e'' - e''').
\end{aligned} \tag{289}$$

It can be checked that $\{\pm e \pm e' \pm e'' \pm e'''\}^2 = e + e' + e'' + e''' = 1$.

3.9 Representation of hyperbolic fourcomplex numbers by irreducible matrices

If T is the unitary matrix,

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \tag{290}$$

it can be shown that the matrix TUT^{-1} has the form

$$TUT^{-1} = \begin{pmatrix} x+y+z+t & 0 & 0 & 0 \\ 0 & x-y+z-t & 0 & 0 \\ 0 & 0 & x+y-z-t & 0 \\ 0 & 0 & 0 & x-y-z+t \end{pmatrix}, \tag{291}$$

where U is the matrix in Eq. (220) used to represent the hyperbolic fourcomplex number u. The relations between the variables $x+y+z+t, x-y+z-t, x+y-z-t, x-y-z+t$ for the multiplication of hyperbolic fourcomplex numbers have been written in Eqs. (204)-(207). The matrix TUT^{-1} provides an irreducible representation [8] of the hyperbolic fourcomplex number u in terms of matrices with real coefficients.

4 Planar Complex Numbers in Four Dimensions

4.1 Operations with planar fourcomplex numbers

A planar fourcomplex number is determined by its four components (x, y, z, t). The sum of the planar fourcomplex numbers (x, y, z, t) and (x', y', z', t') is the planar fourcomplex number $(x+x', y+y', z+z', t+t')$. The product of the planar fourcomplex numbers (x, y, z, t) and (x', y', z', t') is defined in this work to be the planar fourcomplex number $(xx'-yt'-zz'-ty', xy'+yx'-zt'-tz', xz'+yy'+zx'-tt', xt'+yz'+zy'+tx')$.

Planar fourcomplex numbers and their operations can be represented by writing the planar fourcomplex number (x, y, z, t) as $u = x+\alpha y+\beta z+\gamma t$, where α, β and γ are bases for which the multiplication rules are

$$\alpha^2 = \beta,\ \beta^2 = -1,\ \gamma^2 = -\beta, \alpha\beta = \beta\alpha = \gamma,\ \alpha\gamma = \gamma\alpha = -1,\ \beta\gamma = \gamma\beta = -\alpha. \tag{292}$$

Two planar fourcomplex numbers $u = x+\alpha y+\beta z+\gamma t, u' = x'+\alpha y'+\beta z'+\gamma t'$ are equal, $u = u'$, if and only if $x = x', y = y', z = z', t = t'$. If $u = x+\alpha y+\beta z+\gamma t, u' = x'+\alpha y'+\beta z'+\gamma t'$ are planar fourcomplex numbers, the sum $u+u'$ and the product uu' defined above can be obtained by applying the usual algebraic rules to the sum $(x+\alpha y+\beta z+\gamma t)+(x'+\alpha y'+\beta z'+\gamma t')$ and to the product $(x+\alpha y+\beta z+\gamma t)(x'+\alpha y'+\beta z'+\gamma t')$, and grouping of the resulting terms,

$$u+u' = x+x'+\alpha(y+y')+\beta(z+z')+\gamma(t+t'), \tag{293}$$

$$\begin{aligned} uu' = {} & xx'-yt'-zz'-ty'+\alpha(xy'+yx'-zt'-tz')+\beta(xz'+yy'+zx'-tt') \\ & +\gamma(xt'+yz'+zy'+tx'). \end{aligned} \tag{294}$$

If u, u', u'' are planar fourcomplex numbers, the multiplication is associative

$$(uu')u'' = u(u'u'') \tag{295}$$

and commutative

$$uu' = u'u, \tag{296}$$

as can be checked through direct calculation. The planar fourcomplex zero is $0 + \alpha \cdot 0 + \beta \cdot 0 + \gamma \cdot 0$, denoted simply 0, and the planar fourcomplex unity is $1 + \alpha \cdot 0 + \beta \cdot 0 + \gamma \cdot 0$, denoted simply 1.

The inverse of the planar fourcomplex number $u = x + \alpha y + \beta z + \gamma t$ is a planar fourcomplex number $u' = x' + \alpha y' + \beta z' + \gamma t'$ having the property that

$$uu' = 1. \tag{297}$$

Written on components, the condition, Eq. (297), is

$$\begin{aligned} &xx' - ty' - zz' - yt' = 1, \\ &yx' + xy' - tz' - zt' = 0, \\ &zx' + yy' + xz' - tt' = 0, \\ &tx' + zy' + yz' + xt' = 0. \end{aligned} \tag{298}$$

The system (298) has the solution

$$x' = \frac{x(x^2 + z^2) - z(y^2 - t^2) + 2xyt}{\rho^4}, \tag{299}$$

$$y' = -\frac{y(x^2 - z^2) + t(y^2 + t^2) + 2xzt}{\rho^4}, \tag{300}$$

$$z' = \frac{-z(x^2 + z^2) + x(y^2 - t^2) + 2zyt}{\rho^4}, \tag{301}$$

$$t' = -\frac{t(x^2 - z^2) + y(y^2 + t^2) - 2xyz}{\rho^4}, \tag{302}$$

provided that $\rho \neq 0$, where

$$\rho^4 = x^4 + z^4 + y^4 + t^4 + 2x^2z^2 + 2y^2t^2 + 4x^2yt - 4xy^2z + 4xzt^2 - 4yz^2t. \tag{303}$$

The quantity ρ will be called amplitude of the planar fourcomplex number $x + \alpha y + \beta z + \gamma t$. Since

$$\rho^4 = \rho_+^2 \rho_-^2, \tag{304}$$

where

$$\rho_+^2 = \left(x + \frac{y-t}{\sqrt{2}}\right)^2 + \left(z + \frac{y+t}{\sqrt{2}}\right)^2, \ \rho_-^2 = \left(x - \frac{y-t}{\sqrt{2}}\right)^2 + \left(z - \frac{y+t}{\sqrt{2}}\right)^2, \tag{305}$$

a planar fourcomplex number $u = x + \alpha y + \beta z + \gamma t$ has an inverse, unless

$$x + \frac{y-t}{\sqrt{2}} = 0, \ z + \frac{y+t}{\sqrt{2}} = 0, \tag{306}$$

or

$$x - \frac{y-t}{\sqrt{2}} = 0, \ z - \frac{y+t}{\sqrt{2}} = 0. \tag{307}$$

Because of conditions (306)-(307) these 2-dimensional hypersurfaces will be called nodal hyperplanes. It can be shown that if $uu' = 0$ then either $u = 0$, or $u' = 0$, or one of the planar fourcomplex numbers is of the form $x + \alpha(x+z)/\sqrt{2} + \beta z - \gamma(x-z)/\sqrt{2}$ and the other of the form $x' - \alpha(x'+z')/\sqrt{2} + \beta z' + \gamma(x'-z')/\sqrt{2}$.

4.2 Geometric representation of planar fourcomplex numbers

The planar fourcomplex number $x + \alpha y + \beta z + \gamma t$ can be represented by the point A of coordinates (x, y, z, t). If O is the origin of the four-dimensional space x, y, z, t, the distance from A to the origin O can be taken as

$$d^2 = x^2 + y^2 + z^2 + t^2. \tag{308}$$

The distance d will be called modulus of the planar fourcomplex number $x + \alpha y + \beta z + \gamma t$, $d = |u|$. The orientation in the four-dimensional space of the line OA can be specified with the aid of three angles ϕ, χ, ψ defined with respect to the rotated system of axes

$$\xi = \frac{x}{\sqrt{2}} + \frac{y-t}{2}, \ \tau = \frac{x}{\sqrt{2}} - \frac{y-t}{2}, \ \upsilon = \frac{z}{\sqrt{2}} + \frac{y+t}{2}, \ \zeta = -\frac{z}{\sqrt{2}} + \frac{y+t}{2}. \tag{309}$$

The variables $\xi, \upsilon, \tau, \zeta$ will be called canonical planar fourcomplex variables. The use of the rotated axes $\xi, \upsilon, \tau, \zeta$ for the definition of the angles ϕ, χ, ψ is convenient for the expression of the planar fourcomplex numbers in exponential and trigonometric forms, as it will be discussed further. The angle ϕ is the angle between the projection of A in the plane ξ, υ and the $O\xi$ axis, $0 \le \phi < 2\pi$, χ is the angle between the projection of A in the plane τ, ζ and the $O\tau$ axis, $0 \le \chi < 2\pi$, and ψ is the angle between the line OA and the plane $\tau O \zeta$,

$0 \le \psi \le \pi/2$, as shown in Fig. 1. The definition of the variables in this section is different from the definition used for the circular fourcomplex numbers, because the definition of the rotated axes in Eq. (309) is different from the definition of the rotated circular axes, Eq. (18). The angles ϕ and χ will be called azimuthal angles, the angle ψ will be called planar angle . The fact that $0 \le \psi \le \pi/2$ means that ψ has the same sign on both faces of the two-dimensional hyperplane $\upsilon O\zeta$. The components of the point A in terms of the distance d and the angles ϕ, χ, ψ are thus

$$\frac{x}{\sqrt{2}}+\frac{y-t}{2}=d\cos\phi\sin\psi, \tag{310}$$

$$\frac{x}{\sqrt{2}}-\frac{y-t}{2}=d\cos\chi\cos\psi, \tag{311}$$

$$\frac{z}{\sqrt{2}}+\frac{y+t}{2}=d\sin\phi\sin\psi, \tag{312}$$

$$-\frac{z}{\sqrt{2}}+\frac{y+t}{2}=d\sin\chi\cos\psi. \tag{313}$$

It can be checked that $\rho_+=\sqrt{2}d\sin\psi, \rho_-=\sqrt{2}d\cos\psi$. The coordinates x, y, z, t in terms of the variables d, ϕ, χ, ψ are

$$x=\frac{d}{\sqrt{2}}(\cos\phi\sin\psi+\cos\chi\cos\psi), \tag{314}$$

$$y=\frac{d}{\sqrt{2}}[\sin(\phi+\pi/4)\sin\psi+\sin(\chi-\pi/4)\cos\psi], \tag{315}$$

$$z=\frac{d}{\sqrt{2}}(\sin\phi\sin\psi-\sin\chi\cos\psi), \tag{316}$$

$$t=\frac{d}{\sqrt{2}}[-\cos(\phi+\pi/4)\sin\psi+\cos(\chi-\pi/4)\cos\psi]. \tag{317}$$

The angles ϕ, χ, ψ can be expressed in terms of the coordinates x, y, z, t as

$$\sin\phi=\frac{z+(y+t)/\sqrt{2}}{\rho_+},\ \cos\phi=\frac{x+(y-t)/\sqrt{2}}{\rho_+}, \tag{318}$$

$$\sin\chi=\frac{-z+(y+t)/\sqrt{2}}{\rho_-},\ \cos\chi=\frac{x-(y-t)/\sqrt{2}}{\rho_-}, \tag{319}$$

$$\tan\psi=\rho_+/\rho_-. \tag{320}$$

The nodal hyperplanes are $\xi O\upsilon$, for which $\tau=0,\zeta=0$, and $\tau O\zeta$, for which $\xi=0,\upsilon=0$. For points in the nodal hyperplane $\xi O\upsilon$ the planar angle is $\psi=\pi/2$, for points in the nodal hyperplane $\tau O\zeta$ the planar angle is $\psi=0$.

It can be shown that if $u_1=x_1+\alpha y_1+\beta z_1+\gamma t_1, u_2=x_2+\alpha y_2+\beta z_2+\gamma t_2$ are planar fourcomplex numbers of amplitudes and angles $\rho_1,\phi_1,\chi_1,\psi_1$ and respectively $\rho_2,\phi_2,\chi_2,\psi_2$, then the amplitude ρ and the angles ϕ,χ,ψ of the product planar fourcomplex number u_1u_2 are

$$\rho=\rho_1\rho_2, \tag{321}$$

$$\phi=\phi_1+\phi_2,\ \chi=\chi_1+\chi_2,\ \tan\psi=\tan\psi_1\tan\psi_2. \tag{322}$$

The relations (321)-(322) are consequences of the definitions (303)-(305), (318)-(320) and of the identities

$$\begin{aligned}&\left[(x_1x_2-z_1z_2-y_1t_2-t_1y_2)+\frac{(x_1y_2+y_1x_2-z_1t_2-t_1z_2)-(x_1t_2+t_1x_2+z_1y_2+y_1z_2)}{\sqrt{2}}\right]^2\\&+\left[(x_1z_2+z_1x_2+y_1y_2-t_1t_2)+\frac{(x_1y_2+y_1x_2-z_1t_2-t_1z_2)+(x_1t_2+t_1x_2+z_1y_2+y_1z_2)}{\sqrt{2}}\right]^2\\&=\left[\left(x_1+\frac{y_1-t_1}{\sqrt{2}}\right)^2+\left(z_1+\frac{y_1+t_1}{\sqrt{2}}\right)^2\right]\left[\left(x_2+\frac{y_2-t_2}{\sqrt{2}}\right)^2+\left(z_2+\frac{y_2+t_2}{\sqrt{2}}\right)^2\right],\end{aligned} \tag{323}$$

$$\begin{aligned}&\left[(x_1x_2-z_1z_2-y_1t_2-t_1y_2)-\frac{(x_1y_2+y_1x_2-z_1t_2-t_1z_2)-(x_1t_2+t_1x_2+z_1y_2+y_1z_2)}{\sqrt{2}}\right]^2\\&+\left[(x_1z_2+z_1x_2+y_1y_2-t_1t_2)-\frac{(x_1y_2+y_1x_2-z_1t_2-t_1z_2)+(x_1t_2+t_1x_2+z_1y_2+y_1z_2)}{\sqrt{2}}\right]^2\\&=\left[\left(x_1-\frac{y_1-t_1}{\sqrt{2}}\right)^2+\left(z_1-\frac{y_1+t_1}{\sqrt{2}}\right)^2\right]\left[\left(x_2-\frac{y_2-t_2}{\sqrt{2}}\right)^2+\left(z_2-\frac{y_2+t_2}{\sqrt{2}}\right)^2\right],\end{aligned} \tag{324}$$

$$\begin{aligned}&(x_1x_2-z_1z_2-y_1t_2-t_1y_2)+\frac{(x_1y_2+y_1x_2-z_1t_2-t_1z_2)-(x_1t_2+t_1x_2+z_1y_2+y_1z_2)}{\sqrt{2}}\\&=\left(x_1+\frac{y_1-t_1}{\sqrt{2}}\right)\left(x_2+\frac{y_2-t_2}{\sqrt{2}}\right)-\left(z_1+\frac{y_1+t_1}{\sqrt{2}}\right)\left(z_2+\frac{y_2+t_2}{\sqrt{2}}\right),\end{aligned} \tag{325}$$

$$\begin{aligned}&(x_1z_2+z_1x_2+y_1y_2-t_1t_2)+\frac{(x_1y_2+y_1x_2-z_1t_2-t_1z_2)+(x_1t_2+t_1x_2+z_1y_2+y_1z_2)}{\sqrt{2}}\\&=\left(z_1+\frac{y_1+t_1}{\sqrt{2}}\right)\left(x_2+\frac{y_2-t_2}{\sqrt{2}}\right)+\left(x_1+\frac{y_1-t_1}{\sqrt{2}}\right)\left(z_2+\frac{y_2+t_2}{\sqrt{2}}\right),\end{aligned} \tag{326}$$

$$(x_1x_2 - z_1z_2 - y_1t_2 - t_1y_2) - \frac{(x_1y_2 + y_1x_2 - z_1t_2 - t_1z_2) - (x_1t_2 + t_1x_2 + z_1y_2 + y_1z_2)}{\sqrt{2}}$$
$$= \left(x_1 - \frac{y_1 - t_1}{\sqrt{2}}\right)\left(x_2 - \frac{y_2 - t_2}{\sqrt{2}}\right) - \left(-z_1 + \frac{y_1 + t_1}{\sqrt{2}}\right)\left(-z_2 + \frac{y_2 + t_2}{\sqrt{2}}\right), \tag{327}$$

$$-(x_1z_2 + z_1x_2 + y_1y_2 - t_1t_2) + \frac{(x_1y_2 + y_1x_2 - z_1t_2 - t_1z_2) + (x_1t_2 + t_1x_2 + z_1y_2 + y_1z_2)}{\sqrt{2}}$$
$$= \left(-z_1 + \frac{y_1 + t_1}{\sqrt{2}}\right)\left(x_2 - \frac{y_2 - t_2}{\sqrt{2}}\right) + \left(x_1 - \frac{y_1 - t_1}{\sqrt{2}}\right)\left(-z_2 + \frac{y_2 + t_2}{\sqrt{2}}\right). \tag{328}$$

The identities (323) and (324) can also be written as

$$\rho_+^2 = \rho_{1+}\rho_{2+}, \tag{329}$$

$$\rho_-^2 = \rho_{1-}\rho_{2-}, \tag{330}$$

where

$$\rho_{j+}^2 = \left(x_j + \frac{y_j - t_j}{\sqrt{2}}\right)^2 + \left(z_j + \frac{y_j + t_j}{\sqrt{2}}\right)^2, \rho_{j-}^2 = \left(x_j - \frac{y_j - t_j}{\sqrt{2}}\right)^2 + \left(z - \frac{y_j + t_j}{\sqrt{2}}\right)^2, \tag{331}$$

for $j = 1, 2$.

The fact that the amplitude of the product is equal to the product of the amplitudes, as written in Eq. (321), can be demonstrated also by using a representation of the multiplication of the planar fourcomplex numbers by matrices, in which the planar fourcomplex number $u = x + \alpha y + \beta z + \gamma t$ is represented by the matrix

$$A = \begin{pmatrix} x & y & z & t \\ -t & x & y & z \\ -z & -t & x & y \\ -y & -z & -t & x \end{pmatrix}. \tag{332}$$

The product $u = x + \alpha y + \beta z + \gamma t$ of the planar fourcomplex numbers $u_1 = x_1 + \alpha y_1 + \beta z_1 + \gamma t_1, u_2 = x_2 + \alpha y_2 + \beta z_2 + \gamma t_2$, can be represented by the matrix multiplication

$$A = A_1 A_2. \tag{333}$$

It can be checked that the determinant $\det(A)$ of the matrix A is

$$\det A = \rho^4. \tag{334}$$

The identity (321) is then a consequence of the fact the determinant of the product of matrices is equal to the product of the determinants of the factor matrices.

4.3 The planar fourdimensional cosexponential functions

The exponential function of a fourcomplex variable u and the addition theorem for the exponential function have been written in Eqs. (44) and (45). If $u = x + \alpha y + \beta z + \gamma t$, then $\exp u$ can be calculated as $\exp u = \exp x \cdot \exp(\alpha y) \cdot \exp(\beta z) \cdot \exp(\gamma t)$. According to Eq. (292),

$$
\begin{aligned}
&\alpha^{8m} = 1, \alpha^{8m+1} = \alpha, \alpha^{8m+2} = \beta, \alpha^{8m+3} = \gamma, \\
&\alpha^{8m+4} = -1, \alpha^{8m+5} = -\alpha, \alpha^{8m+6} = -\beta, \alpha^{8m+7} = -\gamma, \\
&\beta^{4m} = 1, \beta^{4m+1} = \beta, \beta^{4m+2} = -1, \beta^{4m+3} = -\beta, \\
&\gamma^{8m} = 1, \gamma^{8m+1} = \gamma, \gamma^{8m+2} = -\beta, \gamma^{8m+3} = \alpha, \\
&\gamma^{8m+4} = -1, \gamma^{8m+5} = -\gamma, \gamma^{8m+6} = \beta, \gamma^{8m+7} = -\alpha,
\end{aligned}
\tag{335}
$$

where n is a natural number, so that $\exp(\alpha y)$, $\exp(\beta z)$ and $\exp(\gamma t)$ can be written as

$$\exp(\beta z) = \cos z + \beta \sin z, \tag{336}$$

and

$$\exp(\alpha y) = f_{40}(y) + \alpha f_{41}(y) + \beta f_{42}(y) + \gamma f_{43}(y), \tag{337}$$

$$\exp(\gamma t) = f_{40}(t) + \gamma f_{41}(t) - \beta f_{42}(t) + \alpha f_{43}(t), \tag{338}$$

where the four-dimensional cosexponential functions $f_{40}, f_{41}, f_{42}, f_{43}$ are defined by the series

$$f_{40}(x) = 1 - x^4/4! + x^8/8! - \cdots, \tag{339}$$

$$f_{41}(x) = x - x^5/5! + x^9/9! - \cdots, \tag{340}$$

$$f_{42}(x) = x^2/2! - x^6/6! + x^{10}/10! - \cdots, \tag{341}$$

$$f_{43}(x) = x^3/3! - x^7/7! + x^{11}/11! - \cdots. \tag{342}$$

The functions f_{40}, f_{42} are even, the functions f_{41}, f_{43} are odd,

$$f_{40}(-u) = f_{40}(u),\ f_{42}(-u) = f_{42}(u),\ f_{41}(-u) = -f_{41}(u),\ f_{43}(-u) = -f_{43}(u). \tag{343}$$

Addition theorems for the four-dimensional cosexponential functions can be obtained from the relation $\exp \alpha(x+y) = \exp \alpha x \cdot \exp \alpha y$, by substituting the expression of the exponentials as given in Eq. (337),

$$f_{40}(x+y) = f_{40}(x)f_{40}(y) - f_{41}(x)f_{43}(y) - f_{42}(x)f_{42}(y) - f_{43}(x)f_{41}(y), \tag{344}$$

$$f_{41}(x+y) = f_{40}(x)f_{41}(y) + f_{41}(x)f_{40}(y) - f_{42}(x)f_{43}(y) - f_{43}(x)f_{42}(y), \tag{345}$$

$$f_{42}(x+y) = f_{40}(x)f_{42}(y) + f_{41}(x)f_{41}(y) + f_{42}(x)f_{40}(y) - f_{43}(x)f_{43}(y), \tag{346}$$

$$f_{43}(x+y) = f_{40}(x)f_{43}(y) + f_{41}(x)f_{42}(y) + f_{42}(x)f_{41}(y) + f_{43}(x)f_{40}(y). \tag{347}$$

For $x = y$ the relations (344)-(347) take the form

$$f_{40}(2x) = f_{40}^2(x) - f_{42}^2(x) - 2f_{41}(x)f_{43}(x), \tag{348}$$

$$f_{41}(2x) = 2f_{40}(x)f_{41}(x) - 2f_{42}(x)f_{43}(x), \tag{349}$$

$$f_{42}(2x) = f_{41}^2(x) - f_{43}^2(x) + 2f_{40}(x)f_{42}(x), \tag{350}$$

$$f_{43}(2x) = 2f_{40}(x)f_{43}(x) + 2f_{41}(x)f_{42}(x). \tag{351}$$

For $x = -y$ the relations (344)-(347) and (343) yield

$$f_{40}^2(x) - f_{42}^2(x) + 2f_{41}(x)f_{43}(x) = 1, \tag{352}$$

$$f_{41}^2(x) - f_{43}^2(x) - 2f_{40}(x)f_{42}(x) = 0. \tag{353}$$

From Eqs. (336)-(338) it can be shown that, for m integer,

$$(\cos z + \beta \sin z)^m = \cos mz + \beta \sin mz, \tag{354}$$

and

$$[f_{40}(y)+\alpha f_{41}(y)+\beta f_{42}(y)+\gamma f_{43}(y)]^m = f_{40}(my)+\alpha f_{41}(my)+\beta f_{42}(my)+\gamma f_{43}(my), \tag{355}$$

$$[f_{40}(t)+\gamma f_{41}(t)-\beta f_{42}(t)+\alpha f_{43}(t)]^m = f_{40}(mt)+\gamma f_{41}(mt)-\beta f_{42}(mt)+\alpha f_{43}(mt). \tag{356}$$

Since

$$(\alpha-\gamma)^{2m} = 2^m,\ (\alpha-\gamma)^{2m+1} = 2^m(\alpha-\gamma), \tag{357}$$

it can be shown from the definition of the exponential function, Eq. (44) that

$$\exp(\alpha-\gamma)x = \cosh\sqrt{2}x + \frac{\alpha-\gamma}{\sqrt{2}}\sinh\sqrt{2}x. \tag{358}$$

Substituting in the relation $\exp(\alpha-\gamma)x = \exp\alpha x \exp(-\gamma x)$ the expression of the exponentials from Eqs. (337), (338) and (358) yields

$$f_{40}^2 + f_{41}^2 + f_{42}^2 + f_{43}^2 = \cosh\sqrt{2}x, \tag{359}$$

$$f_{40}f_{41} - f_{40}f_{43} + f_{41}f_{42} + f_{42}f_{43} = \frac{1}{\sqrt{2}}\sinh\sqrt{2}x, \tag{360}$$

where $f_{40}, f_{41}, f_{42}, f_{43}$ are functions of x. From relations (359) and (360) it can be inferred that

$$\left(f_{40} + \frac{f_{41}-f_{43}}{\sqrt{2}}\right)^2 + \left(f_{42} + \frac{f_{41}+f_{43}}{\sqrt{2}}\right)^2 = \exp\sqrt{2}x, \tag{361}$$

$$\left(f_{40} - \frac{f_{41}-f_{43}}{\sqrt{2}}\right)^2 + \left(f_{42} - \frac{f_{41}+f_{43}}{\sqrt{2}}\right)^2 = \exp(-\sqrt{2}x), \tag{362}$$

which means that

$$\left[\left(f_{40} + \frac{f_{41}-f_{43}}{\sqrt{2}}\right)^2 + \left(f_{42} + \frac{f_{41}+f_{43}}{\sqrt{2}}\right)^2\right]\left[\left(f_{40} - \frac{f_{41}-f_{43}}{\sqrt{2}}\right)^2 + \left(f_{42} - \frac{f_{41}+f_{43}}{\sqrt{2}}\right)^2\right] = 1. \tag{363}$$

An equivalent form of Eq. (363) is

$$f_{40}^4+f_{41}^4+f_{42}^4+f_{43}^4+2(f_{40}^2f_{42}^2+f_{41}^2f_{43}^2)+4(f_{40}^2f_{41}f_{43}+f_{40}f_{42}f_{43}^2-f_{40}f_{41}^2f_{42}-f_{41}f_{42}^2f_{43}) = 1. \tag{364}$$

The form of this relation is similar to the expression in Eq. (303). Similarly, since

$$(\alpha+\gamma)^{2m} = (-1)^m 2^m,\ (\alpha+\gamma)^{2m+1} = (-1)^m 2^m(\alpha+\gamma), \tag{365}$$

it can be shown from the definition of the exponential function, Eq. (44) that

$$\exp(\alpha+\gamma)x = \cos\sqrt{2}x + \frac{\alpha+\gamma}{\sqrt{2}}\sin\sqrt{2}x. \tag{366}$$

Substituting in the relation $\exp(\alpha+\gamma)x = \exp\alpha x \exp\gamma x$ the expression of the exponentials from Eqs. (337), (338) and (366) yields

$$f_{40}^2 - f_{41}^2 + f_{42}^2 - f_{43}^2 = \cos\sqrt{2}x, \tag{367}$$

$$f_{40}f_{41} + f_{40}f_{43} - f_{41}f_{42} + f_{42}f_{43} = \frac{1}{\sqrt{2}}\sin\sqrt{2}x, \tag{368}$$

where $f_{40}, f_{41}, f_{42}, f_{43}$ are functions of x.

Expressions of the four-dimensional cosexponential functions (339)-(342) can be obtained using the fact that $[(1+i)/\sqrt{2}]^4 = -1$, so that

$$f_{40}(x) = \frac{1}{2}\left(\cosh\frac{1+i}{\sqrt{2}}x + \cos\frac{1+i}{\sqrt{2}}x\right), \tag{369}$$

$$f_{41}(x) = \frac{1}{\sqrt{2}(1+i)}\left(\sinh\frac{1+i}{\sqrt{2}}x + \sin\frac{1+i}{\sqrt{2}}x\right), \tag{370}$$

$$f_{42}(x) = \frac{1}{2i}\left(\cosh\frac{1+i}{\sqrt{2}}x - \cos\frac{1+i}{\sqrt{2}}x\right), \tag{371}$$

$$f_{43}(x) = \frac{1}{\sqrt{2}(-1+i)}\left(\sinh\frac{1+i}{\sqrt{2}}x - \sin\frac{1+i}{\sqrt{2}}x\right). \tag{372}$$

Using the addition theorems for the functions in the right-hand sides of Eqs. (369)-(372), the expressions of the four-dimensional cosexponential functions become

$$f_{40}(x) = \cos\frac{x}{\sqrt{2}}\cosh\frac{x}{\sqrt{2}}, \tag{373}$$

$$f_{41}(x) = \frac{1}{\sqrt{2}}\left(\sin\frac{x}{\sqrt{2}}\cosh\frac{x}{\sqrt{2}} + \sinh\frac{x}{\sqrt{2}}\cos\frac{x}{\sqrt{2}}\right), \tag{374}$$

$$f_{42}(x) = \sin\frac{x}{\sqrt{2}}\sinh\frac{x}{\sqrt{2}}, \tag{375}$$

$$f_{43}(x) = \frac{1}{\sqrt{2}}\left(\sin\frac{x}{\sqrt{2}}\cosh\frac{x}{\sqrt{2}} - \sinh\frac{x}{\sqrt{2}}\cos\frac{x}{\sqrt{2}}\right). \tag{376}$$

It is remarkable that the series in Eqs. (373)-(376), in which the terms are either of the form x^{4m}, or x^{4m+1}, or x^{4m+2}, or x^{4m+3} can be expressed in terms of elementary functions whose power series are not subject to such restrictions. The graphs of the four-dimensional cosexponential functions are shown in Fig. 3.

It can be checked that the cosexponential functions are solutions of the fourth-order differential equation

$$\frac{\mathrm{d}^4\zeta}{\mathrm{d}u^4} = -\zeta, \tag{377}$$

whose solutions are of the form $\zeta(u) = Af_{40}(u) + Bf_{41}(u) + Cf_{42}(u) + Df_{43}(u)$. It can also be checked that the derivatives of the cosexponential functions are related by

$$\frac{df_{40}}{du} = -f_{43},\ \frac{df_{41}}{du} = f_{40},\ \frac{df_{42}}{du} = f_{41},\ \frac{df_{43}}{du} = f_{42}. \tag{378}$$

4.4 The exponential and trigonometric forms of planar fourcomplex numbers

Any planar fourcomplex number $u = x + \alpha y + \beta z + \gamma t$ can be writen in the form

$$x + \alpha y + \beta z + \gamma t = e^{x_1 + \alpha y_1 + \beta z_1 + \gamma t_1}. \tag{379}$$

The expressions of x_1, y_1, z_1, t_1 as functions of x, y, z, t can be obtained by developing $e^{\alpha y_1}, e^{\beta z_1}$ and $e^{\gamma t_1}$ with the aid of Eqs. (336)-(338), by multiplying these expressions and separating the hypercomplex components, and then substituting the expressions of the four-dimensional cosexponential functions, Eqs. (373)-(376),

$$x = e^{x_1}\left(\cos z_1 \cos\frac{y_1+t_1}{\sqrt{2}}\cosh\frac{y_1-t_1}{\sqrt{2}} - \sin z_1 \sin\frac{y_1+t_1}{\sqrt{2}}\sinh\frac{y_1-t_1}{\sqrt{2}}\right), \tag{380}$$

$$y = e^{x_1}\left[\sin\left(z_1+\frac{\pi}{4}\right)\cos\frac{y_1+t_1}{\sqrt{2}}\sinh\frac{y_1-t_1}{\sqrt{2}} + \cos\left(z_1+\frac{\pi}{4}\right)\sin\frac{y_1+t_1}{\sqrt{2}}\cosh\frac{y_1-t_1}{\sqrt{2}}\right], \tag{381}$$

$$z = e^{x_1}\left(\cos z_1 \sin\frac{y_1+t_1}{\sqrt{2}}\sinh\frac{y_1-t_1}{\sqrt{2}} + \sin z_1 \cos\frac{y_1+t_1}{\sqrt{2}}\cosh\frac{y_1-t_1}{\sqrt{2}}\right), \tag{382}$$

$$t = e^{x_1}\left[-\cos\left(z_1+\frac{\pi}{4}\right)\cos\frac{y_1+t_1}{\sqrt{2}}\sinh\frac{y_1-t_1}{\sqrt{2}} + \sin\left(z_1+\frac{\pi}{4}\right)\sin\frac{y_1+t_1}{\sqrt{2}}\cosh\frac{y_1-t_1}{\sqrt{2}}\right], \tag{383}$$

The relations (380)-(383) can be rewritten as

$$x + \frac{y-t}{\sqrt{2}} = e^{x_1}\cos\left(z_1+\frac{y_1+t_1}{\sqrt{2}}\right)e^{(y_1-t_1)/\sqrt{2}}, \tag{384}$$

$$z + \frac{y+t}{\sqrt{2}} = e^{x_1}\sin\left(z_1+\frac{y_1+t_1}{\sqrt{2}}\right)e^{(y_1-t_1)/\sqrt{2}}, \tag{385}$$

$$x - \frac{y-t}{\sqrt{2}} = e^{x_1}\cos\left(z_1-\frac{y_1+t_1}{\sqrt{2}}\right)e^{-(y_1-t_1)/\sqrt{2}}, \tag{386}$$

$$z - \frac{y+t}{\sqrt{2}} = e^{x_1}\sin\left(z_1-\frac{y_1+t_1}{\sqrt{2}}\right)e^{-(y_1-t_1)/\sqrt{2}}. \tag{387}$$

By multiplying the sum of the squares of the first two and of the last two relations (384)-(387) it results that

$$e^{4x_1} = \rho_+^2\rho_-^2, \tag{388}$$

or

$$e^{x_1} = \rho. \tag{389}$$

By summing the squares of all relations (384)-(387) it results that

$$d^2=\rho^2\cosh\left[\sqrt{2}(y_1-t_1)\right]. \tag{390}$$

Then the quantities y_1, z_1, t_1 can be expressed in terms of the angles ϕ, χ, ψ defined in Eqs. (310)-(313) as

$$z_1+\frac{y_1+t_1}{\sqrt{2}}=\phi, \tag{391}$$

$$-z_1+\frac{y_1+t_1}{\sqrt{2}}=\chi, \tag{392}$$

$$\frac{e^{(y_1-t_1)/\sqrt{2}}}{\sqrt{2}\left[\cosh\sqrt{2}(y_1-t_1)\right]^{1/2}}=\sin\psi,\quad \frac{e^{-(y_1-t_1)/\sqrt{2}}}{\sqrt{2}\left[\cosh\sqrt{2}(y_1-t_1)\right]^{1/2}}=\cos\psi. \tag{393}$$

From Eq. (393) it results that

$$y_1-t_1=\frac{1}{\sqrt{2}}\ln\tan\psi, \tag{394}$$

so that

$$y_1=\frac{\phi+\chi}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\ln\tan\psi,\ z_1=\frac{\phi-\chi}{2},\ t_1=\frac{\phi+\chi}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}\ln\tan\psi. \tag{395}$$

Substituting the expressions of the quantities x_1, y_1, z_1, t_1 in Eq. (379) yields

$$u=\rho\exp\left[\frac{1}{2\sqrt{2}}(\alpha-\gamma)\ln\tan\psi+\frac{1}{2}\left(\beta+\frac{\alpha+\gamma}{\sqrt{2}}\right)\phi-\frac{1}{2}\left(\beta-\frac{\alpha+\gamma}{\sqrt{2}}\right)\chi\right], \tag{396}$$

which will be called the exponential form of the planar fourcomplex number u. It can be checked that

$$\exp\left[\frac{1}{2}\left(\beta+\frac{\alpha+\gamma}{\sqrt{2}}\right)\phi\right]=\frac{1}{2}-\frac{\alpha-\gamma}{2\sqrt{2}}+\left(\frac{1}{2}+\frac{\alpha-\gamma}{2\sqrt{2}}\right)\cos\phi+\left(\frac{\beta}{2}+\frac{\alpha+\gamma}{2\sqrt{2}}\right)\sin\phi, \tag{397}$$

$$\exp\left[-\frac{1}{2}\left(\beta-\frac{\alpha+\gamma}{\sqrt{2}}\right)\chi\right]=\frac{1}{2}+\frac{\alpha-\gamma}{2\sqrt{2}}+\left(\frac{1}{2}-\frac{\alpha-\gamma}{2\sqrt{2}}\right)\cos\chi-\left(\frac{\beta}{2}-\frac{\alpha+\gamma}{2\sqrt{2}}\right)\sin\chi, \tag{398}$$

which shows that $e^{[\beta+(\alpha+\gamma)/\sqrt{2}]\phi/2}$ and $e^{-[\beta-(\alpha+\gamma)/\sqrt{2}]\chi/2}$ are periodic functions of ϕ and respectively χ, with period 2π.

The exponential of the logarithmic term in Eq. (396) can be expanded with the aid of the relation (358) as

$$\exp\left[\frac{1}{2\sqrt{2}}(\alpha-\gamma)\ln\tan\psi\right]=\frac{1}{(\sin 2\psi)^{1/2}}\left[\cos\left(\psi-\frac{\pi}{4}\right)+\frac{\alpha-\gamma}{\sqrt{2}}\sin\left(\psi-\frac{\pi}{4}\right)\right]. \tag{399}$$

Since according to Eq. (320) $\tan\psi=\rho_+/\rho_-$, then

$$\sin\psi\cos\psi=\frac{\rho_+\rho_-}{\rho_+^2+\rho_-^2}, \tag{400}$$

and it can be checked that

$$\rho_+^2+\rho_-^2=2d^2, \tag{401}$$

where d has been defined in Eq. (308). Thus

$$\rho^2=d^2\sin 2\psi, \tag{402}$$

so that the planar fourcomplex number u can be written as

$$u=d\left[\cos\left(\psi-\frac{\pi}{4}\right)+\frac{\alpha-\gamma}{\sqrt{2}}\sin\left(\psi-\frac{\pi}{4}\right)\right]\exp\left[\frac{1}{2}\left(\beta+\frac{\alpha+\gamma}{\sqrt{2}}\right)\phi-\frac{1}{2}\left(\beta-\frac{\alpha+\gamma}{\sqrt{2}}\right)\chi\right], \tag{403}$$

which will be called the trigonometric form of the planar fourcomplex number u.

If u_1,u_2 are planar fourcomplex numbers of moduli and angles d_1,ϕ_1,χ_1,ψ_1 and respectively d_2,ϕ_2,χ_2,ψ_2, the product of the factors depending on the planar angles can be calculated to be

$$\begin{aligned}&[\cos(\psi_1-\pi/4)+\frac{\alpha-\gamma}{\sqrt{2}}\sin(\psi_1-\pi/4)][\cos(\psi_2-\pi/4)+\frac{\alpha-\gamma}{\sqrt{2}}\sin(\psi_2-\pi/4)]\\&\quad=[\cos(\psi_1-\psi_2)-\frac{\alpha-\gamma}{\sqrt{2}}\cos(\psi_1+\psi_2)].\end{aligned} \tag{404}$$

The right-hand side of Eq. (404) can be written as

$$\begin{aligned}&\cos(\psi_1-\psi_2)-\frac{\alpha-\gamma}{\sqrt{2}}\cos(\psi_1+\psi_2)\\&\quad=[2(\cos^2\psi_1\cos^2\psi_2+\sin^2\psi_1\sin^2\psi_2)]^{1/2}[\cos(\psi-\pi/4)+\frac{\alpha-\gamma}{\sqrt{2}}\sin(\psi-\pi/4)],\end{aligned} \tag{405}$$

where the angle ψ, determined by the condition that

$$\tan(\psi-\pi/4)=-\cos(\psi_1+\psi_2)/\cos(\psi_1-\psi_2) \tag{406}$$

is given by $\tan\psi=\tan\psi_1\tan\psi_2$, which is consistent with Eq. (322). The modulus d of the product u_1u_2 is then

$$d=\sqrt{2}d_1d_2\left(\cos^2\psi_1\cos^2\psi_2+\sin^2\psi_1\sin^2\psi_2\right)^{1/2}. \tag{407}$$

4.5 Elementary functions of planar fourcomplex variables

The logarithm u_1 of the planar fourcomplex number u, $u_1 = \ln u$, can be defined as the solution of the equation

$$u = e^{u_1}, \tag{408}$$

written explicitly previously in Eq. (379), for u_1 as a function of u. From Eq. (396) it results that

$$\ln u = \ln \rho + \frac{1}{2\sqrt{2}}(\alpha - \gamma)\ln\tan\psi + \frac{1}{2}\left(\beta + \frac{\alpha+\gamma}{\sqrt{2}}\right)\phi - \frac{1}{2}\left(\beta - \frac{\alpha+\gamma}{\sqrt{2}}\right)\chi, \tag{409}$$

which is multivalued because of the presence of the terms proportional to ϕ and χ. It can be inferred from Eqs. (321) and (322) that

$$\ln(uu') = \ln u + \ln u', \tag{410}$$

up to multiples of $\pi[\beta + (\alpha+\gamma)/\sqrt{2}]$ and $\pi[\beta - (\alpha+\gamma)/\sqrt{2}]$.

The power function u^m can be defined for real values of n as

$$u^m = e^{m \ln u}. \tag{411}$$

The power function is multivalued unless n is an integer. For integer n, it can be inferred from Eq. (410) that

$$(uu')^m = u^m\, u'^m. \tag{412}$$

If, for example, $m = 2$, it can be checked with the aid of Eq. (403) that Eq. (411) gives indeed $(x + \alpha y + \beta z + \gamma t)^2 = x^2 - z^2 - 2yt + 2\alpha(xy - zt) + \beta(y^2 - t^2 + 2xz) + 2\gamma(xt + yz)$.

The trigonometric functions of the fourcomplex variable u and the addition theorems for these functions have been written in Eqs. (90)-(93). The cosine and sine functions of the hypercomplex variables $\alpha y, \beta z$ and γt can be expressed as

$$\cos\alpha y = f_{40}(y) - \beta f_{42}(y),\ \sin\alpha y = \alpha f_{41}(y) - \gamma f_{43}(y), \tag{413}$$

$$\cos\beta z = \cosh z,\ \sin\beta z = \beta\sinh z, \tag{414}$$

$$\cos\gamma t = f_{40}(t) + \beta f_{42}(t),\ \sin\gamma t = \gamma f_{41}(t) - \alpha f_{43}(t). \tag{415}$$

The cosine and sine functions of a planar fourcomplex number $x+\alpha y+\beta z+\gamma t$ can then be expressed in terms of elementary functions with the aid of the addition theorems Eqs. (92), (93) and of the expressions in Eqs. (413)-(415).

The hyperbolic functions of the fourcomplex variable u and the addition theorems for these functions have been written in Eqs. (97)-(100). The hyperbolic cosine and sine functions of the hypercomplex variables $\alpha y, \beta z$ and γt can be expressed as

$$\cosh\alpha y = f_{40}(y)+\beta f_{42}(y),\ \sinh\alpha y = \alpha f_{41}(y)+\gamma f_{43}(y), \tag{416}$$

$$\cosh\beta z = \cos z,\ \sinh\beta z = \beta\sin z, \tag{417}$$

$$\cosh\gamma t = f_{40}(t)-\beta f_{42}(t),\ \sinh\gamma t = \gamma f_{41}(t)+\alpha f_{43}(t). \tag{418}$$

The hyperbolic cosine and sine functions of a planar fourcomplex number $x+\alpha y+\beta z+\gamma t$ can then be expressed in terms of elementary functions with the aid of the addition theorems Eqs. (99), (100) and of the expressions in Eqs. (416)-(418).

4.6 Power series of planar fourcomplex variables

A planar fourcomplex series is an infinite sum of the form

$$a_0+a_1+a_2+\cdots+a_l+\cdots, \tag{419}$$

where the coefficients a_l are planar fourcomplex numbers. The convergence of the series (419) can be defined in terms of the convergence of its 4 real components. The convergence of a planar fourcomplex series can however be studied using planar fourcomplex variables. The main criterion for absolute convergence remains the comparison theorem, but this requires a number of inequalities which will be discussed further.

The modulus of a planar fourcomplex number $u=x+\alpha y+\beta z+\gamma t$ can be defined as

$$|u| = (x^2+y^2+z^2+t^2)^{1/2}, \tag{420}$$

so that, according to Eq. (308), $d=|u|$. Since $|x|\le|u|, |y|\le|u|, |z|\le|u|, |t|\le|u|$, a property of absolute convergence established via a comparison theorem based on the modulus of the series (419) will ensure the absolute convergence of each real component of that series.

The modulus of the sum u_1+u_2 of the planar fourcomplex numbers u_1, u_2 fulfils the inequality

$$||u_1|-|u_2|| \le |u_1+u_2| \le |u_1|+|u_2|. \tag{421}$$

For the product the relation is

$$|u_1u_2| \le \sqrt{2}|u_1||u_2|, \tag{422}$$

as can be shown from Eq. (407). The relation (422) replaces the relation of equality extant for regular complex numbers. The equality in Eq. (422) takes place for $\cos^2(\psi_1-\psi_2)=1$, $\cos^2(\psi_1+\psi_2)=1$, which means that $x_1+(y_1-t_1)/\sqrt{2}=0$, $z_1+(y_1+t_1)/\sqrt{2}=0$, $x_2+(y_2-t_2)/\sqrt{2}=0$, $z_2+(y_2+t_2)/\sqrt{2}=0$, or $x_1-(y_1-t_1)/\sqrt{2}=0$, $z_1-(y_1+t_1)/\sqrt{2}=0$, $x_2-(y_2-t_2)/\sqrt{2}=0$, $z_2-(y_2+t_2)/\sqrt{2}=0$. The modulus of the product, which has the property that $0\le|u_1u_2|$, becomes equal to zero for $\cos^2(\psi_1-\psi_2)=0$, $\cos^2(\psi_1+\psi_2)=0$, which means that $x_1+(y_1-t_1)/\sqrt{2}=0$, $z_1+(y_1+t_1)/\sqrt{2}=0$, $x_2-(y_2-t_2)/\sqrt{2}=0$, $z_2-(y_2+t_2)/\sqrt{2}=0$, or $x_1-(y_1-t_1)/\sqrt{2}=0$, $z_1-(y_1+t_1)/\sqrt{2}=0$, $x_2+(y_2-t_2)/\sqrt{2}=0$, $z_2+(y_2+t_2)/\sqrt{2}=0$. as discussed after Eq. (307).

It can be shown that

$$x^2+y^2+z^2+t^2 \le |u^2| \le \sqrt{2}(x^2+y^2+z^2+t^2). \tag{423}$$

The left relation in Eq. (423) becomes an equality for $\sin^2 2\psi=1$, when $\rho_+=\rho_-$, which means that $x(y-t)+z(y+t)=0$. The right relation in Eq. (423) becomes an equality for $\sin^2 2\psi=0$, when $x+(y-t)/\sqrt{2}=0$, $z+(y+t)/\sqrt{2}=0$, or $x-(y-t)/\sqrt{2}=0$, $z-(y+t)/\sqrt{2}=0$. The inequality in Eq. (422) implies that

$$|u^l| \le 2^{(l-1)/2}|u|^l. \tag{424}$$

From Eqs. (422) and (424) it results that

$$|au^l| \le 2^{l/2}|a||u|^l. \tag{425}$$

A power series of the planar fourcomplex variable u is a series of the form

$$a_0+a_1u+a_2u^2+\cdots+a_lu^l+\cdots. \tag{426}$$

Since

$$\left|\sum_{l=0}^{\infty} a_l u^l\right| \le \sum_{l=0}^{\infty} 2^{l/2}|a_l||u|^l, \tag{427}$$

a sufficient condition for the absolute convergence of this series is that

$$\lim_{l\to\infty} \frac{\sqrt{2}|a_{l+1}||u|}{|a_l|} < 1. \tag{428}$$

Thus the series is absolutely convergent for

$$|u| < c, \tag{429}$$

where

$$c = \lim_{l\to\infty} \frac{|a_l|}{\sqrt{2}|a_{l+1}|}. \tag{430}$$

The convergence of the series (426) can be also studied with the aid of the transformation

$$x + \alpha y + \beta z + \gamma t = \sqrt{2}(e_1\xi + \tilde{e}_1\upsilon + e_2\tau + \tilde{e}_2\zeta), \tag{431}$$

where $\xi, \upsilon, \tau, \zeta$ have been defined in Eq. (309), and

$$e_1 = \frac{1}{2} + \frac{\alpha-\gamma}{2\sqrt{2}},\ \ \tilde{e}_1 = \frac{\beta}{2} + \frac{\alpha+\gamma}{2\sqrt{2}},\ \ e_2 = \frac{1}{2} - \frac{\alpha-\gamma}{2\sqrt{2}},\ \ \tilde{e}_2 = -\frac{\beta}{2} + \frac{\alpha+\gamma}{2\sqrt{2}}. \tag{432}$$

It can be checked that

$$\begin{gathered} e_1^2 = e_1,\ \ \tilde{e}_1^2 = -e_1,\ \ e_1\tilde{e}_1 = \tilde{e}_1,\ \ e_2^2 = e_2,\ \ \tilde{e}_2^2 = -e_2,\ \ e_2\tilde{e}_2 = \tilde{e}_2, \\ e_1e_2 = 0,\ \ \tilde{e}_1\tilde{e}_2 = 0,\ \ e_1\tilde{e}_2 = 0,\ \ e_2\tilde{e}_1 = 0. \end{gathered} \tag{433}$$

The moduli of the bases in Eq. (432) are

$$|e_1| = \frac{1}{\sqrt{2}},\ |\tilde{e}_1| = \frac{1}{\sqrt{2}},\ |e_2| = \frac{1}{\sqrt{2}},\ |\tilde{e}_2| = \frac{1}{\sqrt{2}}, \tag{434}$$

and it can be checked that

$$|x + \alpha y + \beta z + \gamma t|^2 = \xi^2 + \upsilon^2 + \tau^2 + \zeta^2. \tag{435}$$

The ensemble $e_1, \tilde{e}_1, e_2, \tilde{e}_2$ will be called the canonical planar fourcomplex base, and Eq. (431) gives the canonical form of the planar fourcomplex number.

If $u = u'u''$, the components $\xi, \upsilon, \tau, \zeta$ are related, according to Eqs. (325)-(328) by

$$\xi = \sqrt{2}(\xi'\xi'' - \upsilon'\upsilon''), \upsilon = \sqrt{2}(\xi'\upsilon'' + \upsilon'\xi''), \tau = \sqrt{2}(\tau'\tau'' - \zeta'\zeta''), \zeta = \sqrt{2}(\tau'\zeta'' + \zeta'\xi''), \tag{436}$$

which show that, upon multiplication, the components ξ, υ and τ, ζ obey, up to a normalization constant, the same rules as the real and imaginary components of usual, two-dimensional complex numbers.

If the coefficients in Eq. (426) are

$$a_l = a_{l0} + \alpha a_{l1} + \beta a_{l2} + \gamma a_{l3}, \tag{437}$$

and

$$A_{l1} = a_{l0} + \frac{a_{l1} - a_{l3}}{\sqrt{2}},\ \tilde{A}_{l1} = a_{l2} + \frac{a_{l1} + a_{l3}}{\sqrt{2}},\ A_{l2} = a_{l0} - \frac{a_{l1} - a_{l3}}{\sqrt{2}},\ \tilde{A}_{l2} = -a_{l2} + \frac{a_{l1} + a_{l3}}{\sqrt{2}}, \tag{438}$$

the series (426) can be written as

$$\sum_{l=0}^{\infty} 2^{l/2} \left[(e_1 A_{l1} + \tilde{e}_1 \tilde{A}_{l1})(e_1 \xi + \tilde{e}_1 \upsilon)^l + (e_2 A_{l2} + \tilde{e}_2 \tilde{A}_{l2})(e_2 \tau + \tilde{e}_2 \zeta)^l \right]. \tag{439}$$

Thus, the series in Eqs. (426) and (439) are absolutely convergent for

$$\rho_+ < c_1,\ \rho_- < c_2, \tag{440}$$

where

$$c_1 = \lim_{l\to\infty} \frac{\left[A_{l1}^2 + \tilde{A}_{l1}^2\right]^{1/2}}{\sqrt{2}\left[A_{l+1,1}^2 + \tilde{A}_{l+1,1}^2\right]^{1/2}},\ \ c_2 = \lim_{l\to\infty} \frac{\left[A_{l2}^2 + \tilde{A}_{l2}^2\right]^{1/2}}{\sqrt{2}\left[A_{l+1,2}^2 + \tilde{A}_{l+1,2}^2\right]^{1/2}}. \tag{441}$$

It can be shown that $c = (1/\sqrt{2})\min(c_1, c_2)$, where min designates the smallest of the numbers c_1, c_2. Using the expression of $|u|$ in Eq. (435), it can be seen that the spherical region of convergence defined in Eqs. (429), (430) is included in the cylindrical region of convergence defined in Eqs. (440) and (441).

4.7 Analytic functions of planar fourcomplex variables

The fourcomplex function $f(u)$ of the fourcomplex variable u has been expressed in Eq. (136) in terms of the real functions $P(x,y,z,t), Q(x,y,z,t), R(x,y,z,t), S(x,y,z,t)$ of real variables x, y, z, t. The relations between the partial derivatives of the functions P, Q, R, S are obtained by setting succesively in Eq. (137) $\Delta x \to 0, \Delta y = \Delta z = \Delta t = 0$; then $\Delta y \to 0, \Delta x = \Delta z = \Delta t = 0$; then $\Delta z \to 0, \Delta x = \Delta y = \Delta t = 0$; and finally $\Delta t \to 0, \Delta x = \Delta y = \Delta z = 0$. The relations are

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} = \frac{\partial R}{\partial z} = \frac{\partial S}{\partial t}, \tag{442}$$

$$\frac{\partial Q}{\partial x}=\frac{\partial R}{\partial y}=\frac{\partial S}{\partial z}=-\frac{\partial P}{\partial t}, \tag{443}$$

$$\frac{\partial R}{\partial x}=\frac{\partial S}{\partial y}=-\frac{\partial P}{\partial z}=-\frac{\partial Q}{\partial t}, \tag{444}$$

$$\frac{\partial S}{\partial x}=-\frac{\partial P}{\partial y}=-\frac{\partial Q}{\partial z}=-\frac{\partial R}{\partial t}. \tag{445}$$

The relations (442)-(445) are analogous to the Riemann relations for the real and imaginary components of a complex function. It can be shown from Eqs. (442)-(445) that the component P is a solution of the equations

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}=0,\ \frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial t^2}=0, \tag{446}$$

and the components Q, R, S are solutions of similar equations. As can be seen from Eqs. (446), the components P, Q, R, S of an analytic function of planar fourcomplex variable are harmonic with respect to the pairs of variables x, y and z, t. The component P is also a solution of the mixed-derivative equations

$$\frac{\partial^2 P}{\partial x^2}=-\frac{\partial^2 P}{\partial y\partial t},\ \frac{\partial^2 P}{\partial y^2}=\frac{\partial^2 P}{\partial x\partial z},\ \frac{\partial^2 P}{\partial z^2}=\frac{\partial^2 P}{\partial y\partial t},\ \frac{\partial^2 P}{\partial t^2}=-\frac{\partial^2 P}{\partial x\partial z}, \tag{447}$$

and the components Q, R, S are solutions of similar equations. The component P is also a solution of the mixed-derivative equations

$$\frac{\partial^2 P}{\partial x\partial y}=-\frac{\partial^2 P}{\partial z\partial t},\ \frac{\partial^2 P}{\partial x\partial t}=\frac{\partial^2 P}{\partial y\partial z}, \tag{448}$$

and the components Q, R, S are solutions of similar equations.

4.8 Integrals of functions of planar fourcomplex variables

The singularities of planar fourcomplex functions arise from terms of the form $1/(u-u_0)^m$, with $m>0$. Functions containing such terms are singular not only at $u=u_0$, but also at all points of the two-dimensional hyperplanes passing through u_0 and which are parallel to the nodal hyperplanes.

The integral of a planar fourcomplex function between two points A, B along a path situated in a region free of singularities is independent of path, which means that the integral of an analytic function along a loop situated in a region free from singularities is zero,

$$\oint_\Gamma f(u)du=0, \tag{449}$$

where it is supposed that a surface Σ spanning the closed loop Γ is not intersected by any of the two-dimensional hyperplanes associated with the singularities of the function $f(u)$. Using the expression, Eq. (136) for $f(u)$ and the fact that $du = dx + \alpha dy + \beta dz + \gamma dt$, the explicit form of the integral in Eq. (449) is

$$\oint_\Gamma f(u)du = \oint_\Gamma [(Pdx - Sdy - Rdz - Qdt) + \alpha(Qdx + Pdy - Sdz - Rdt)$$
$$+\beta(Rdx + Qdy + Pdz - Sdt) + \gamma(Sdx + Rdy + Qdz + Pdt)]. \tag{450}$$

If the functions P, Q, R, S are regular on a surface Σ spanning the loop Γ, the integral along the loop Γ can be transformed with the aid of the theorem of Stokes in an integral over the surface Σ of terms of the form $\partial P/\partial y + \partial S/\partial x,\ \partial P/\partial z + \partial R/\partial x,\ \partial P/\partial t + \partial Q/\partial x,\ \partial R/\partial y - \partial S/\partial z,\ \partial S/\partial t - \partial Q/\partial y,\ \partial R/\partial t - \partial Q/\partial z$ and of similar terms arising from the α, β and γ components, which are equal to zero by Eqs. (442)-(445), and this proves Eq. (449).

The integral of the function $(u - u_0)^m$ on a closed loop Γ is equal to zero for m a positive or negative integer not equal to -1,

$$\oint_\Gamma (u - u_0)^m du = 0,\ m \text{ integer},\ m \neq -1. \tag{451}$$

This is due to the fact that $\int (u - u_0)^m du = (u - u_0)^{m+1}/(m+1)$, and to the fact that the function $(u - u_0)^{m+1}$ is singlevalued for m an integer.

The integral $\oint du/(u - u_0)$ can be calculated using the exponential form (396),

$$u - u_0 = \rho \exp\left[\frac{1}{2\sqrt{2}}(\alpha - \gamma)\ln\tan\psi + \frac{1}{2}\left(\beta + \frac{\alpha+\gamma}{\sqrt{2}}\right)\phi - \frac{1}{2}\left(\beta - \frac{\alpha+\gamma}{\sqrt{2}}\right)\chi\right], \tag{452}$$

so that

$$\frac{du}{u - u_0} = \frac{d\rho}{\rho} + \frac{1}{2\sqrt{2}}(\alpha - \gamma)d\ln\tan\psi + \frac{1}{2}\left(\beta + \frac{\alpha+\gamma}{\sqrt{2}}\right)d\phi - \frac{1}{2}\left(\beta - \frac{\alpha+\gamma}{\sqrt{2}}\right)d\chi. \tag{453}$$

Since ρ and ψ are singlevalued variables, it follows that $\oint_\Gamma d\rho/\rho = 0, \oint_\Gamma d\ln\tan\psi = 0$. On the other hand, ϕ and χ are cyclic variables, so that they may give a contribution to the integral around the closed loop Γ. Thus, if C_+ is a circle of radius r parallel to the $\xi O\upsilon$ plane, whose projection of the center of this circle on the $\xi O\upsilon$ plane coincides with the projection of the point u_0 on this plane, the points of the circle C_+ are described according to Eqs. (309)-(313) by the equations

$$\xi = \xi_0 + r\sin\psi\cos\phi,\ \upsilon = \upsilon_0 + r\sin\psi\sin\phi,\ \tau = \tau_0 + r\cos\psi\cos\chi,$$
$$\zeta = \zeta_0 + r\cos\psi\sin\chi, \tag{454}$$

for constant values of χ and ψ, $\psi \neq 0, \pi/2$, where $u_0 = x_0 + \alpha y_0 + \beta z_0 + \gamma t_0$, and $\xi_0, \upsilon_0, \tau_0, \zeta_0$ are calculated from x_0, y_0, z_0, t_0 according to Eqs. (309). Then

$$\oint_{C_+} \frac{du}{u-u_0} = \pi\left(\beta + \frac{\alpha+\gamma}{\sqrt{2}}\right). \tag{455}$$

If C_- is a circle of radius r parallel to the $\tau O\zeta$ plane, whose projection of the center of this circle on the $\tau O\zeta$ plane coincides with the projection of the point u_0 on this plane, the points of the circle C_- are described by the same Eqs. (454) but for constant values of ϕ and ψ, $\psi \neq 0, \pi/2$. Then

$$\oint_{C_-} \frac{du}{u-u_0} = -\pi\left(\beta - \frac{\alpha+\gamma}{\sqrt{2}}\right). \tag{456}$$

The expression of $\oint_\Gamma du/(u-u_0)$ can be written as a single equation with the aid of the functional $\mathrm{int}(M,C)$ defined in Eq. (153) as

$$\oint_\Gamma \frac{du}{u-u_0} = \pi\left(\beta + \frac{\alpha+\gamma}{\sqrt{2}}\right)\ \mathrm{int}(u_{0\xi\upsilon}, \Gamma_{\xi\upsilon}) - \pi\left(\beta - \frac{\alpha+\gamma}{\sqrt{2}}\right)\ \mathrm{int}(u_{0\tau\zeta}, \Gamma_{\tau\zeta}), \tag{457}$$

where $u_{0\xi\upsilon}, u_{0\tau\zeta}$ and $\Gamma_{\xi\upsilon}, \Gamma_{\tau\zeta}$ are respectively the projections of the point u_0 and of the loop Γ on the planes $\xi\upsilon$ and $\tau\zeta$.

If $f(u)$ is an analytic planar fourcomplex function which can be expanded in a series as written in Eq. (131), and the expansion holds on the curve Γ and on a surface spanning Γ, then from Eqs. (451) and (457) it follows that

$$\oint_\Gamma \frac{f(u)du}{u-u_0} = \pi\left[\left(\beta + \frac{\alpha+\gamma}{\sqrt{2}}\right)\ \mathrm{int}(u_{0\xi\upsilon}, \Gamma_{\xi\upsilon}) - \left(\beta - \frac{\alpha+\gamma}{\sqrt{2}}\right)\ \mathrm{int}(u_{0\tau\zeta}, \Gamma_{\tau\zeta})\right]\ f(u_0), \tag{458}$$

where $\Gamma_{\xi\upsilon}, \Gamma_{\tau\zeta}$ are the projections of the curve Γ on the planes $\xi\upsilon$ and respectively $\tau\zeta$, as shown in Fig. 2. As remarked previously, the definition of the variables in this section is different from the former definition for the circular hypercomplex numbers.

Substituting in the right-hand side of Eq. (458) the expression of $f(u)$ in terms of the real components P, Q, R, S, Eq. (136), yields

$$\begin{aligned}
\oint_\Gamma \frac{f(u)du}{u-u_0}
&= \pi\left[\left(\beta + \frac{\alpha+\gamma}{\sqrt{2}}\right)P - \left(1 + \frac{\alpha-\gamma}{\sqrt{2}}\right)R - \left(\gamma - \frac{1-\beta}{\sqrt{2}}\right)Q - \left(\alpha - \frac{1+\beta}{\sqrt{2}}\right)S\right]\mathrm{int}\left(u_{0\xi\upsilon}, \Gamma_{\xi\upsilon}\right)\\
&-\pi\left[\left(\beta - \frac{\alpha+\gamma}{\sqrt{2}}\right)P - \left(1 - \frac{\alpha-\gamma}{\sqrt{2}}\right)R - \left(\gamma - \frac{1+\beta}{\sqrt{2}}\right)Q - \left(\alpha - \frac{1-\beta}{\sqrt{2}}\right)S\right]\mathrm{int}(u_{0\tau\zeta}, \Gamma_{\tau\zeta}),
\end{aligned} \tag{459}$$

where P, Q, R, S are the values of the components of f at $u = u_0$.

If $f(u)$ can be expanded as written in Eq. (131) on Γ and on a surface spanning Γ, then from Eqs. (451) and (457) it also results that

$$\oint_\Gamma \frac{f(u)du}{(u-u_0)^{m+1}} = \frac{\pi}{m!}\left[\left(\beta+\frac{\alpha+\gamma}{\sqrt{2}}\right)\,\mathrm{int}(u_{0\xi\upsilon},\Gamma_{\xi\upsilon})-\left(\beta-\frac{\alpha+\gamma}{\sqrt{2}}\right)\,\mathrm{int}(u_{0\tau\zeta},\Gamma_{\tau\zeta})\right]\,f^{(m)}(u_0), \quad (460)$$

where it has been used the fact that the derivative $f^{(m)}(u_0)$ of order n of $f(u)$ at $u = u_0$ is related to the expansion coefficient in Eq. (131) according to Eq. (135).

If a function $f(u)$ is expanded in positive and negative powers of $u - u_j$, where u_j are planar fourcomplex constants, j being an index, the integral of f on a closed loop Γ is determined by the terms in the expansion of f which are of the form $a_j/(u - u_j)$,

$$f(u) = \cdots + \sum_j \frac{a_j}{u-u_j} + \cdots \quad (461)$$

Then the integral of f on a closed loop Γ is

$$\oint_\Gamma f(u)du = \pi\left(\beta+\frac{\alpha+\gamma}{\sqrt{2}}\right)\sum_j \mathrm{int}(u_{j\xi\upsilon},\Gamma_{\xi\upsilon})a_j - \pi\left(\beta-\frac{\alpha+\gamma}{\sqrt{2}}\right)\sum_j \mathrm{int}(u_{j\tau\zeta},\Gamma_{\tau\zeta})a_j. \quad (462)$$

4.9 Factorization of planar fourcomplex polynomials

A polynomial of degree m of the planar fourcomplex variable $u = x + \alpha y + \beta z + \gamma t$ has the form

$$P_m(u) = u^m + a_1 u^{m-1} + \cdots + a_{m-1}u + a_m, \quad (463)$$

where the constants are in general planar fourcomplex numbers.

It can be shown that any planar fourcomplex polynomial has a planar fourcomplex root, whence it follows that a polynomial of degree m can be written as a product of m linear factors of the form $u - u_j$, where the planar fourcomplex numbers u_j are the roots of the polynomials, although the factorization may not be unique,

$$P_m(u) = \prod_{j=1}^{m}(u - u_j). \quad (464)$$

The fact that any planar fourcomplex polynomial has a root can be shown by considering the transformation of a fourdimensional sphere with the center at the origin by the function

u^m. The points of the hypersphere of radius d are of the form written in Eq. (403), with d constant and ϕ, χ, ψ arbitrary. The point u^m is

$$u^m = d^m \left[\cos\left(\psi - \frac{\pi}{4}\right) + \frac{\alpha-\gamma}{\sqrt{2}} \sin\left(\psi - \frac{\pi}{4}\right)\right]^m \exp\left[\frac{1}{2}\left(\beta + \frac{\alpha+\gamma}{\sqrt{2}}\right) m\phi - \frac{1}{2}\left(\beta - \frac{\alpha+\gamma}{\sqrt{2}}\right) m\chi\right]. \tag{465}$$

It can be shown with the aid of Eq. (407) that

$$\left| u \exp\left[\frac{1}{2}\left(\beta + \frac{\alpha+\gamma}{\sqrt{2}}\right)\phi - \frac{1}{2}\left(\beta - \frac{\alpha+\gamma}{\sqrt{2}}\right)\chi\right]\right| = |u|, \tag{466}$$

so that

$$\left|\left[\cos(\psi - \pi/4) + \frac{\alpha-\gamma}{\sin}(\psi - \pi/4)\right]^m \exp\left[\frac{1}{2}\left(\beta + \frac{\alpha+\gamma}{\sqrt{2}}\right) m\phi - \frac{1}{2}\left(\beta - \frac{\alpha+\gamma}{\sqrt{2}}\right) m\chi\right]\right| = \left|\left(\cos(\psi - \pi/4) + \frac{\alpha-\gamma}{\sqrt{2}} \sin(\psi - \pi/4)\right)^m\right|. \tag{467}$$

The right-hand side of Eq. (467) is

$$\left|\left(\cos\epsilon + \frac{\alpha-\gamma}{\sqrt{2}}\sin\epsilon\right)^m\right|^2 = \sum_{k=0}^{m} C_{2m}^{2k} \cos^{2m-2k}\epsilon \sin^{2k}\epsilon, \tag{468}$$

where $\epsilon = \psi - \pi/4$, and since $C_{2m}^{2k} \geq C_m^k$, it can be concluded that

$$\left|\left(\cos\epsilon + \frac{\alpha-\gamma}{\sqrt{2}}\sin\epsilon\right)^m\right|^2 \geq 1. \tag{469}$$

Then

$$d^m \leq |u^m| \leq 2^{(m-1)/2} d^m, \tag{470}$$

which shows that the image of a four-dimensional sphere via the transformation operated by the function u^m is a finite hypersurface.

If $u' = u^m$, and

$$u' = d'\left[\cos(\psi' - \pi/4) + \frac{\alpha-\gamma}{\sqrt{2}}\sin(\psi' - \pi/4)\right] \exp\left[\frac{1}{2}\left(\beta + \frac{\alpha+\gamma}{\sqrt{2}}\right)\phi' - \frac{1}{2}\left(\beta - \frac{\alpha+\gamma}{\sqrt{2}}\right)\chi'\right], \tag{471}$$

then

$$\phi' = m\phi,\ \chi' = m\chi,\ \tan\psi' = \tan^m\psi. \tag{472}$$

Since for any values of the angles ϕ', χ', ψ' there is a set of solutions ϕ, χ, ψ of Eqs. (472), and since the image of the hypersphere is a finite hypersurface, it follows that the image of

the four-dimensional sphere via the function u^m is also a closed hypersurface. A continuous hypersurface is called closed when any ray issued from the origin intersects that surface at least once in the finite part of the space.

A transformation of the four-dimensional space by the polynomial $P_m(u)$ will be considered further. By this transformation, a hypersphere of radius d having the center at the origin is changed into a certain finite closed surface, as discussed previously. The transformation of the four-dimensional space by the polynomial $P_m(u)$ associates to the point $u = 0$ the point $f(0) = a_m$, and the image of a hypersphere of very large radius d can be represented with good approximation by the image of that hypersphere by the function u^m. The origin of the axes is an inner point of the latter image. If the radius of the hypersphere is now reduced continuously from the initial very large values to zero, the image hypersphere encloses initially the origin, but the image shrinks to a_m when the radius approaches the value zero. Thus, the origin is initially inside the image hypersurface, and it lies outside the image hypersurface when the radius of the hypersphere tends to zero. Then since the image hypersurface is closed, the image surface must intersect at some stage the origin of the axes, which means that there is a point u_1 such that $f(u_1) = 0$. The factorization in Eq. (464) can then be obtained by iterations.

The roots of the polynomial P_m can be obtained by the following method. If the constants in Eq. (463) are $a_l = a_{l0} + \alpha a_{l1} + \beta a_{l2} + \gamma a_{l3}$, and with the notations of Eq. (438), the polynomial $P_m(u)$ can be written as

$$\begin{aligned} P_m = &\sum_{l=0}^{m} 2^{(m-l)/2}(e_1 A_{l1} + \tilde{e}_1 \tilde{A}_{l1})(e_1\xi + \tilde{e}_1\upsilon)^{m-l} \\ &+ \sum_{l=0}^{m} 2^{(m-l)/2}(e_2 A_{l2} + \tilde{e}_2 \tilde{A}_{l2})(e_2\tau + \tilde{e}_2\zeta)^{m-l}, \end{aligned} \tag{473}$$

where the constants $A_{lk}, \tilde{A}_{lk}, k = 1, 2$ are real numbers. Each of the polynomials of degree m in $e_1\xi + \tilde{e}_1\upsilon, e_2\tau + \tilde{e}_2\zeta$ in Eq. (473) can always be written as a product of linear factors of the form $e_1(\xi - \xi_p) + \tilde{e}_1(\upsilon - \upsilon_p)$ and respectively $e_2(\tau - \tau_p) + \tilde{e}_2(\zeta - \zeta_p)$, where the constants $\xi_p, \upsilon_p, \tau_p, \zeta_p$ are real,

$$\sum_{l=0}^{m} 2^{(m-l)/2}(e_1 A_{l1} + \tilde{e}_1 \tilde{A}_{l1})(e_1\xi + \tilde{e}_1\upsilon)^{m-l} = \prod_{p=1}^{m} 2^{m/2}\left\{e_1(\xi - \xi_p) + \tilde{e}_1(\upsilon - \upsilon_p)\right\}, \tag{474}$$

$$\sum_{l=0}^{m} 2^{(m-l)/2}(e_2A_{l2}+\tilde{e}_2\tilde{A}_{l2})(e_2\tau+\tilde{e}_2\zeta)^{m-l} = \prod_{p=1}^{m} 2^{m/2}\left\{e_2(\tau-\tau_p)+\tilde{e}_2(\zeta-\zeta_p)\right\}. \quad (475)$$

Due to the relations (433), the polynomial $P_m(u)$ can be written as a product of factors of the form

$$P_m(u) = \prod_{p=1}^{m} 2^{m/2}\left\{e_1(\xi-\xi_p)+\tilde{e}_1(\upsilon-\upsilon_p)+e_2(\tau-\tau_p)+\tilde{e}_2(\zeta-\zeta_p)\right\}. \quad (476)$$

This relation can be written with the aid of Eq. (431) in the form (464), where

$$u_p = \sqrt{2}(e_1\xi_p+\tilde{e}_1\upsilon_p+e_2\tau_p+\tilde{e}_2\zeta_p). \quad (477)$$

The roots $e_1\xi_p+\tilde{e}_1\upsilon_p$ and $e_2\tau_p+\tilde{e}_2\zeta_p$ defined in Eqs. (474) and respectively (475) may be ordered arbitrarily. This means that Eq. (477) gives sets of m roots $u_1, ..., u_m$ of the polynomial $P_m(u)$, corresponding to the various ways in which the roots $e_1\xi_p+\tilde{e}_1\upsilon_p$ and $e_2\tau_p+\tilde{e}_2\zeta_p$ are ordered according to p for each polynomial. Thus, while the hypercomplex components in Eqs. (474), (475) taken separately have unique factorizations, the polynomial $P_m(u)$ can be written in many different ways as a product of linear factors. The result of the planar fourcomplex integration, Eq. (462), is however unique.

If, for example, $P(u)=u^2+1$, the possible factorizations are $P=(u-\tilde{e}_1-\tilde{e}_2)(u+\tilde{e}_1+\tilde{e}_2)$ and $P=(u-\tilde{e}_1+\tilde{e}_2)(u+\tilde{e}_1-\tilde{e}_2)$ which can also be written as $u^2+1=(u-\beta)(u+\beta)$ or as $u^2+1=\left\{u-(\alpha+\gamma)/\sqrt{2}\right\}\left\{(u+(\alpha+\gamma)/\sqrt{2}\right\}$. The result of the planar fourcomplex integration, Eq. (462), is however unique. It can be checked that $(\pm\tilde{e}_1\pm\tilde{e}_2)^2=-e_1-e_2=-1$.

4.10 Representation of planar fourcomplex numbers by irreducible matrices

If T is the unitary matrix,

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}, \quad (478)$$

it can be shown that the matrix TUT^{-1} has the form

$$TUT^{-1} = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}, \quad (479)$$

where U is the matrix in Eq. (332) used to represent the planar fourcomplex number u. In Eq. (479), V_1, V_2 are the matrices

$$V_1 = \begin{pmatrix} x + \frac{y-t}{\sqrt{2}} & z + \frac{y+t}{\sqrt{2}} \\ -z - \frac{y+t}{\sqrt{2}} & x + \frac{y-t}{\sqrt{2}} \end{pmatrix}, \quad V_2 = \begin{pmatrix} x - \frac{y-t}{\sqrt{2}} & -z + \frac{y+t}{\sqrt{2}} \\ z - \frac{y+t}{\sqrt{2}} & x - \frac{y-t}{\sqrt{2}} \end{pmatrix}. \tag{480}$$

In Eq. (479), the symbols 0 denote the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{481}$$

The relations between the variables $x+(y-t)/\sqrt{2}, z+(y+t)/\sqrt{2}, x-(y-t)/\sqrt{2}, -z+(y+t)/\sqrt{2}$ for the multiplication of planar fourcomplex numbers have been written in Eqs. (325)-(328). The matrix TUT^{-1} provides an irreducible representation [8] of the planar fourcomplex number u in terms of matrices with real coefficients.

5 Polar omplex Numbers in Four Dimensions

5.1 Operations with polar fourcomplex numbers

A polar fourcomplex number is determined by its four components (x, y, z, t). The sum of the polar fourcomplex numbers (x, y, z, t) and (x', y', z', t') is the polar fourcomplex number $(x + x', y + y', z + z', t + t')$. The product of the polar fourcomplex numbers (x, y, z, t) and (x', y', z', t') is defined in this work to be the polar fourcomplex number $(xx' + yt' + zz' + ty', xy' + yx' + zt' + tz', xz' + yy' + zx' + tt', xt' + yz' + zy' + tx')$. Polar fourcomplex numbers and their operations can be represented by writing the polar fourcomplex number (x, y, z, t) as $u = x + \alpha y + \beta z + \gamma t$, where α, β and γ are bases for which the multiplication rules are

$$\alpha^2 = \beta,\ \beta^2 = 1,\ \gamma^2 = \beta, \alpha\beta = \beta\alpha = \gamma,\ \alpha\gamma = \gamma\alpha = -1,\ \beta\gamma = \gamma\beta = \alpha. \tag{482}$$

Two polar fourcomplex numbers $u = x+\alpha y+\beta z+\gamma t, u' = x'+\alpha y'+\beta z'+\gamma t'$ are equal, $u = u'$, if and only if $x = x', y = y', z = z', t = t'$. If $u = x+\alpha y+\beta z+\gamma t, u' = x'+\alpha y'+\beta z'+\gamma t'$ are polar fourcomplex numbers, the sum $u+u'$ and the product uu' defined above can be obtained by applying the usual algebraic rules to the sum $(x + \alpha y + \beta z + \gamma t) + (x' + \alpha y' + \beta z' + \gamma t')$

and to the product $(x+\alpha y+\beta z+\gamma t)(x'+\alpha y'+\beta z'+\gamma t')$, and grouping of the resulting terms,

$$u+u'=x+x'+\alpha(y+y')+\beta(z+z')+\gamma(t+t'), \tag{483}$$

$$\begin{aligned} uu' = & xx'+yt'+zz'+ty'+\alpha(xy'+yx'+zt'+tz')+\beta(xz'+yy'+zx'+tt') \\ & +\gamma(xt'+yz'+zy'+tx'). \end{aligned} \tag{484}$$

If u, u', u'' are polar fourcomplex numbers, the multiplication is associative

$$(uu')u''=u(u'u'') \tag{485}$$

and commutative

$$uu'=u'u, \tag{486}$$

as can be checked through direct calculation. The polar fourcomplex zero is $0+\alpha\cdot 0+\beta\cdot 0+\gamma\cdot 0$, denoted simply 0, and the polar fourcomplex unity is $1+\alpha\cdot 0+\beta\cdot 0+\gamma\cdot 0$, denoted simply 1.

The inverse of the polar fourcomplex number $u=x+\alpha y+\beta z+\gamma t$ is a polar fourcomplex number $u'=x'+\alpha y'+\beta z'+\gamma t'$ having the property that

$$uu'=1. \tag{487}$$

Written on components, the condition, Eq. (487), is

$$\begin{aligned} xx'+ty'+zz'+yt'&=1,\\ yx'+xy'+tz'+zt'&=0,\\ zx'+yy'+xz'+tt'&=0,\\ tx'+zy'+yz'+xt'&=0. \end{aligned} \tag{488}$$

The system (488) has the solution

$$x'=\frac{x(x^2-z^2)+z(y^2+t^2)-2xyt}{\nu}, \tag{489}$$

$$y'=\frac{-y(x^2+z^2)+t(y^2-t^2)+2xzt}{\nu}, \tag{490}$$

$$z'=\frac{-z(x^2-z^2)+x(y^2+t^2)-2yzt}{\nu}, \tag{491}$$

$$t' = \frac{-t(x^2+z^2)-y(y^2-t^2)+2xyz}{\nu}, \tag{492}$$

provided that $\nu \neq 0$, where

$$\nu = x^4+z^4-y^4-t^4-2x^2z^2+2y^2t^2-4x^2yt-4yz^2t+4xy^2z+4xzt^2. \tag{493}$$

The quantity ν can be written as

$$\nu = v_+ v_- \mu_+^2, \tag{494}$$

where

$$v_+ = x+y+z+t,\ v_- = x-y+z-t, \tag{495}$$

and

$$\mu_+^2 = (x-z)^2+(y-t)^2. \tag{496}$$

Then a polar fourcomplex number $q = x+\alpha y+\beta z+\gamma t$ has an inverse, unless

$$v_+ = 0,\ \text{ or }\ v_- = 0,\ \text{ or }\ \mu_+ = 0. \tag{497}$$

The condition $v_+ = 0$ represents the 3-dimensional hyperplane $x+y+z+t=0$, the condition $v_- = 0$ represents the 3-dimensional hyperplane $x-y+z-t=0$, and the condition $\mu_+ = 0$ represents the 2-dimensional hyperplane $x=z, y=t$. For arbitrary values of the variables x, y, z, t, the quantity ν can be positive or negative. If $\nu \geq 0$, the quantity $\rho = \nu^{1/4}$ will be called amplitude of the polar fourcomplex number $x+\alpha y+\beta z+\gamma t$. Because of conditions (497), these hyperplanes will be called nodal hyperplanes.

It can be shown that if $uu' = 0$ then either $u = 0$, or $u' = 0$, or the polar fourcomplex numbers u, u' belong to different members of the pairs of orthogonal hypersurfaces listed further,

$$x+y+z+t=0\ \text{ and }\ x'=y'=z'=t', \tag{498}$$

$$x-y+z-t=0\ \text{ and }\ x'=-y'=z'=-t'. \tag{499}$$

Divisors of zero also exist if the polar fourcomplex numbers u, u' belong to different members of the pair of two-dimensional hypersurfaces,

$$x-z=0,\ y-t=0\ \text{ and }\ x'+z'=0,\ y'+t'=0. \tag{500}$$

5.2 Geometric representation of polar fourcomplex numbers

The polar fourcomplex number $x+\alpha y+\beta z+\gamma t$ can be represented by the point A of coordinates (x, y, z, t). If O is the origin of the four-dimensional space x, y, z, t, the distance from A to the origin O can be taken as

$$d^2 = x^2 + y^2 + z^2 + t^2. \tag{501}$$

The distance d will be called modulus of the polar fourcomplex number $x+\alpha y+\beta z+\gamma t$, $d = |u|$.

If $u = x+\alpha y+\beta z+\gamma t, u_1 = x_1+\alpha y_1+\beta z_1+\gamma t_1, u_2 = x_2+\alpha y_2+\beta z_2+\gamma t_2$, and $u = u_1u_2$, and if

$$s_{j+} = x_j + y_j + z_j + t_j,\ s_{j-} = x_j - y_j + z_j - t_j, \tag{502}$$

for $j = 1, 2$, it can be shown that

$$v_+ = s_{1+}s_{2+},\ \ v_- = s_{1-}s_{2-}. \tag{503}$$

The relations (503) are a consequence of the identities

$$\begin{aligned}
&(x_1x_2 + z_1z_2 + t_1y_2 + y_1t_2) + (x_1y_2 + y_1x_2 + z_1t_2 + t_1z_2)\\
&\quad+(x_1z_2 + z_1x_2 + y_1y_2 + t_1t_2) + (x_1t_2 + t_1x_2 + z_1y_2 + y_1z_2)\\
&\quad= (x_1 + y_1 + z_1 + t_1)(x_2 + y_2 + z_2 + t_2),
\end{aligned} \tag{504}$$

$$\begin{aligned}
&(x_1x_2 + z_1z_2 + t_1y_2 + y_1t_2) - (x_1y_2 + y_1x_2 + z_1t_2 + t_1z_2)\\
&\quad+(x_1z_2 + z_1x_2 + y_1y_2 + t_1t_2) - (x_1t_2 + t_1x_2 + z_1y_2 + y_1z_2)\\
&\quad= (x_1 + z_1 - y_1 - t_1)(x_2 + z_2 - y_2 - t_2).
\end{aligned} \tag{505}$$

The differences

$$v_1 = x - z,\ \tilde{v}_1 = y - t \tag{506}$$

can be written with the aid of the radius μ_+, Eq. (496), and of the azimuthal angle ϕ, where $0 \le \phi < 2\pi$ as

$$v_1 = \mu_+ \cos\phi,\ \ \tilde{v}_1 = \mu_+ \sin\phi. \tag{507}$$

The variables $v_+, v_-, v_1, \tilde v_1$ will be called canonical polar fourcomplex variables. The distance d, Eq. (501), can then be written as

$$d^2=\frac{1}{4}v_+^2+\frac{1}{4}v_-^2+\frac{1}{2}\mu_+^2. \tag{508}$$

It can be shown that if $u_1=x_1+\alpha y_1+\beta z_1+\gamma t_1, u_2=x_2+\alpha y_2+\beta z_2+\gamma t_2$ are polar fourcomplex numbers of polar radii and angles ρ_{1-}, ϕ_1 and respectively ρ_{2-}, ϕ_2, then the polar radius ρ and the angle ϕ of the product polar fourcomplex number u_1u_2 are

$$\mu_+=\rho_{1-}\rho_{2-}, \tag{509}$$

$$\phi=\phi_1+\phi_2. \tag{510}$$

The relation (509) is a consequence of the identity

$$\begin{aligned}&[(x_1x_2+z_1z_2+y_1t_2+t_1y_2)-(x_1z_2+z_1x_2+y_1y_2+t_1t_2)]^2\\&\quad+[(x_1y_2+y_1x_2+z_1t_2+t_1z_2)-(x_1t_2+t_1x_2+z_1y_2+y_1z_2)]^2\\&\quad=\left[(x_1-z_1)^2+(y_1-t_1)^2\right]\left[(x_2-z_2)^2+(y_2-t_2)^2\right],\end{aligned} \tag{511}$$

and the relation (510) is a consequence of the identities

$$\begin{aligned}&(x_1x_2+z_1z_2+y_1t_2+t_1y_2)-(x_1z_2+z_1x_2+y_1y_2+t_1t_2)\\&\quad=(x_1-z_1)(x_2-z_2)-(y_1-t_1)(y_2-t_2),\end{aligned} \tag{512}$$

$$\begin{aligned}&(x_1y_2+y_1x_2+z_1t_2+t_1z_2)-(x_1t_2+t_1x_2+z_1y_2+y_1z_2)\\&\quad=(y_1-t_1)(x_2-z_2)+(x_1-z_1)(y_2-t_2).\end{aligned} \tag{513}$$

A consequence of Eqs. (503) and (509) is that if $u=u_1u_2$, and $\nu_j=s_js_j''\rho_{j-}$, where $j=1,2$, then

$$\nu=\nu_1\nu_2. \tag{514}$$

The angles θ_+, θ_- between the line OA and the v_+ and respectively v_- axes are

$$\tan\theta_+=\frac{\sqrt{2}\mu_+}{v_+}, \tan\theta_-=\frac{\sqrt{2}\mu_+}{v_-}, \tag{515}$$

where $0 \leq \theta_+ \leq \pi,\ 0 \leq \theta_- \leq \pi$. The variable μ_+ can be expressed with the aid of Eq. (508) as

$$\mu_+^2 = 2d^2\left(1+\frac{1}{\tan^2\theta_+}+\frac{1}{\tan^2\theta_-}\right)^{-1}. \tag{516}$$

The coordinates x, y, z, t can then be expressed in terms of the distance d, of the polar angles θ_+, θ_- and of the azimuthal angle ϕ as

$$x = \frac{\mu_+(\tan\theta_+ + \tan\theta_-)}{2\sqrt{2}\tan\theta_+\tan\theta_-} + \frac{1}{2}\mu_+\cos\phi, \tag{517}$$

$$y = \frac{\mu_+(-\tan\theta_+ + \tan\theta_-)}{2\sqrt{2}\tan\theta_+\tan\theta_-} + \frac{1}{2}\mu_+\sin\phi, \tag{518}$$

$$z = \frac{\mu_+(\tan\theta_+ + \tan\theta_-)}{2\sqrt{2}\tan\theta_+\tan\theta_-} - \frac{1}{2}\mu_+\cos\phi, \tag{519}$$

$$t = \frac{\mu_+(-\tan\theta_+ + \tan\theta_-)}{2\sqrt{2}\tan\theta_+\tan\theta_-} - \frac{1}{2}\mu_+\sin\phi. \tag{520}$$

If $u = u_1u_2$, then Eqs. (503) and (509) imply that

$$\tan\theta_+ = \frac{1}{\sqrt{2}}\tan\theta_{1+}\tan\theta_{2+},\ \tan\theta_- = \frac{1}{\sqrt{2}}\tan\theta_{1-}\tan\theta_{2-}, \tag{521}$$

where

$$\tan\theta_{j+} = \frac{\sqrt{2}\rho_{i+}}{s_j}, \tan\theta_{j-} = \frac{\sqrt{2}\rho_{i-}}{s_j''}. \tag{522}$$

An alternative choice of the angular variables is

$$\mu_+ = \sqrt{2}d\cos\theta,\ v_+ = 2d\sin\theta\cos\lambda,\ v_- = 2d\sin\theta\sin\lambda, \tag{523}$$

where $0 \leq \theta \leq \pi/2, 0 \leq \lambda < 2\pi$. If $u = u_1u_2$, then

$$\tan\lambda = \tan\lambda_1\tan\lambda_2,\ d\cos\theta = \sqrt{2}d_1d_2\cos\theta_1\cos\theta_2, \tag{524}$$

where

$$\rho_{j-} = \sqrt{2}d_j\cos\theta_j,\ s_j = 2d_j\sin\theta_j\cos\lambda_j,\ s_j'' = 2d_j\sin\theta_j\sin\lambda_j, \tag{525}$$

for $j = 1, 2$. The coordinates x, y, z, t can then be expressed in terms of the distance d, of the polar angles θ,λ and of the azimuthal angle ϕ as

$$x = \frac{d}{\sqrt{2}}\sin\theta\sin(\lambda+\pi/4) + \frac{d}{\sqrt{2}}\cos\theta\cos\phi, \tag{526}$$

$$y=\frac{d}{\sqrt{2}}\sin\theta\cos(\lambda+\pi/4)+\frac{d}{\sqrt{2}}\cos\theta\sin\phi, \tag{527}$$

$$z=\frac{d}{\sqrt{2}}\sin\theta\sin(\lambda+\pi/4)-\frac{d}{\sqrt{2}}\cos\theta\cos\phi, \tag{528}$$

$$t=\frac{d}{\sqrt{2}}\sin\theta\cos(\lambda+\pi/4)-\frac{d}{\sqrt{2}}\cos\theta\sin\phi. \tag{529}$$

The polar fourcomplex numbers

$$e_+=\frac{1+\alpha+\beta+\gamma}{4},\ e_-=\frac{1-\alpha+\beta-\gamma}{4}, \tag{530}$$

have the property that

$$e_+^2=e_+,\ e_-^2=e_-,\ e_+e_-=0. \tag{531}$$

The polar fourcomplex numbers

$$e_1=\frac{1-\beta}{2},\tilde{e}_1=\frac{\alpha-\gamma}{2} \tag{532}$$

have the property that

$$e_1^2=e_1,\ \ \tilde{e}_1^2=-e_1,\ \ e_1\tilde{e}_1=\tilde{e}_1. \tag{533}$$

The polar fourcomplex numbers e_+,e_- are orthogonal to $e_1,\tilde{e}_1$,

$$e_+\,e_1=0,\ \ e_+\,\tilde{e}_1=0,\ \ e_-\,e_1=0,\ \ e_-\,\tilde{e}_1=0. \tag{534}$$

The polar fourcomplex number $q=x+\alpha y+\beta z+\gamma t$ can then be written as

$$x+\alpha y+\beta z+\gamma t=v_+e_++v_-e_-+v_1e_1+\tilde{v}_1\tilde{e}_1. \tag{535}$$

The ensemble $e_+,e_-,e_1,\tilde{e}_1$ will be called the canonical polar fourcomplex base, and Eq. (535) gives the canonical form of the polar fourcomplex number. Thus, the product of the polar fourcomplex numbers u,u' can be expressed as

$$uu'=v_+v_+'e_++v_-v_-'e_-+(v_1v_1'-\tilde{v}_1\tilde{v}_1')e_1+(v_1\tilde{v}_1'+v_1'\tilde{v}_1)\tilde{e}_1, \tag{536}$$

where $v_+'=x'+y'+z'+t',v_-'=x'-y'+z'-t',v_1'=x'-y',\tilde{v}_1'=z'-t'$. The moduli of the bases used in Eq. (535) are

$$|e_+|=\frac{1}{2},\ |e_-|=\frac{1}{2},\ |e_1|=\frac{1}{\sqrt{2}},\ |\tilde{e}_1|=\frac{1}{\sqrt{2}}. \tag{537}$$

The fact that the amplitude of the product is equal to the product of the amplitudes, as written in Eq. (514), can be demonstrated also by using a representation of the multiplication of the polar fourcomplex numbers by matrices, in which the polar fourcomplex number $u = x + \alpha y + \beta z + \gamma t$ is represented by the matrix

$$A = \begin{pmatrix} x & y & z & t \\ t & x & y & z \\ z & t & x & y \\ y & z & t & x \end{pmatrix}. \tag{538}$$

The product $u = x + \alpha y + \beta z + \gamma t$ of the polar fourcomplex numbers $u_1 = x_1 + \alpha y_1 + \beta z_1 + \gamma t_1, u_2 = x_2 + \alpha y_2 + \beta z_2 + \gamma t_2$, can be represented by the matrix multiplication

$$A = A_1 A_2. \tag{539}$$

It can be checked that the determinant $\det(A)$ of the matrix A is

$$\det A = \nu. \tag{540}$$

The identity (514) is then a consequence of the fact the determinant of the product of matrices is equal to the product of the determinants of the factor matrices.

5.3 The polar fourdimensional cosexponential functions

The exponential function of a fourcomplex variable u and the addition theorem for the exponential function have been written in Eqs. (44) and (45). If $u = x + \alpha y + \beta z + \gamma t$, then $\exp u$ can be calculated as $\exp u = \exp x \cdot \exp(\alpha y) \cdot \exp(\beta z) \cdot \exp(\gamma t)$. According to Eq. (482),

$$\begin{aligned} &\alpha^{4m} = 1, \alpha^{4m+1} = \alpha, \alpha^{4m+2} = \beta, \alpha^{4m+3} = \gamma, \\ &\beta^{2m} = 1, \beta^{2m+1} = \beta, \\ &\gamma^{4m} = 1, \gamma^{4m+1} = \gamma, \gamma^{4m+2} = \beta, \gamma^{4m+3} = \alpha, \end{aligned} \tag{541}$$

where m is a natural number, so that $\exp(\alpha y)$, $\exp(\beta z)$ and $\exp(\gamma z)$ can be written as

$$\exp(\beta z) = \cosh z + \beta \sinh z, \tag{542}$$

and

$$\exp(\alpha y) = g_{40}(y) + \alpha g_{41}(y) + \beta g_{42}(y) + \gamma g_{43}(y), \tag{543}$$

$$\exp(\gamma t) = g_{40}(t) + \gamma g_{41}(t) + \beta g_{42}(t) + \alpha g_{43}(t), \tag{544}$$

where the four-dimensional cosexponential functions $g_{40}, g_{41}, g_{42}, g_{43}$ are defined by the series

$$g_{40}(x) = 1 + x^4/4! + x^8/8! + \cdots, \tag{545}$$

$$g_{41}(x) = x + x^5/5! + x^9/9! + \cdots, \tag{546}$$

$$g_{42}(x) = x^2/2! + x^6/6! + x^{10}/10! + \cdots, \tag{547}$$

$$g_{43}(x) = x^3/3! + x^7/7! + x^{11}/11! + \cdots. \tag{548}$$

The functions g_{40}, g_{42} are even, the functions g_{41}, g_{43} are odd,

$$g_{40}(-u) = g_{40}(u),\ g_{42}(-u) = g_{42}(u),\ g_{41}(-u) = -g_{41}(u),\ g_{43}(-u) = -g_{43}(u). \tag{549}$$

It can be seen from Eqs. (545)-(548) that

$$g_{40} + g_{41} + g_{42} + g_{43} = e^x,\ g_{40} - g_{41} + g_{42} - g_{43} = e^{-x}, \tag{550}$$

and

$$g_{40} - g_{42} = \cos x,\ g_{41} - g_{43} = \sin x, \tag{551}$$

so that

$$(g_{40} + g_{41} + g_{42} + g_{43})(g_{40} - g_{41} + g_{42} - g_{43}) \left[(g_{40} - g_{42})^2 + (g_{41} - g_{43})^2\right] = 1, \tag{552}$$

which can be also written as

$$g_{40}^4 - g_{41}^4 + g_{42}^4 - g_{43}^4 - 2(g_{40}^2 g_{42}^2 - g_{41}^2 g_{43}^2) - 4(g_{40}^2 g_{41} g_{43} + g_{42}^2 g_{41} g_{43} - g_{41}^2 g_{40} g_{42} - g_{43}^2 g_{40} g_{42}) = 1. \tag{553}$$

The combination of terms in Eq. (553) is similar to that in Eq. (493).

Addition theorems for the four-dimensional cosexponential functions can be obtained from the relation $\exp\alpha(x+y) = \exp\alpha x \cdot \exp\alpha y$, by substituting the expression of the exponentials as given in Eq. (543),

$$g_{40}(x+y) = g_{40}(x)g_{40}(y) + g_{41}(x)g_{43}(y) + g_{42}(x)g_{42}(y) + g_{43}(x)g_{41}(y), \tag{554}$$

$$g_{41}(x+y) = g_{40}(x)g_{41}(y) + g_{41}(x)g_{40}(y) + g_{42}(x)g_{43}(y) + g_{43}(x)g_{42}(y), \tag{555}$$

$$g_{42}(x+y) = g_{40}(x)g_{42}(y) + g_{41}(x)g_{41}(y) + g_{42}(x)g_{40}(y) + g_{43}(x)g_{43}(y), \tag{556}$$

$$g_{43}(x+y) = g_{40}(x)g_{43}(y) + g_{41}(x)g_{42}(y) + g_{42}(x)g_{41}(y) + g_{43}(x)g_{40}(y). \tag{557}$$

For $x = y$ the relations (554)-(557) take the form

$$g_{40}(2x) = g_{40}^2(x) + g_{42}^2(x) + 2g_{41}(x)g_{43}(x), \tag{558}$$

$$g_{41}(2x) = 2g_{40}(x)g_{41}(x) + 2g_{42}(x)g_{43}(x), \tag{559}$$

$$g_{42}(2x) = g_{41}^2(x) + g_{43}^2(x) + 2g_{40}(x)g_{42}(x), \tag{560}$$

$$g_{43}(2x) = 2g_{40}(x)g_{43}(x) + 2g_{41}(x)g_{42}(x). \tag{561}$$

For $x = -y$ the relations (554)-(557) and (549) yield

$$g_{40}^2(x) + g_{42}^2(x) - 2g_{41}(x)g_{43}(x) = 1, \tag{562}$$

$$g_{41}^2(x) + g_{43}^2(x) - 2g_{40}(x)g_{42}(x) = 0. \tag{563}$$

From Eqs. (542)-(544) it can be shown that, for m integer,

$$(\cosh z + \beta \sinh z)^m = \cosh mz + \beta \sinh mz, \tag{564}$$

and

$$[g_{40}(y)+\alpha g_{41}(y)+\beta g_{42}(y)+\gamma g_{43}(y)]^m = g_{40}(my)+\alpha g_{41}(my)+\beta g_{42}(my)+\gamma g_{43}(my), \tag{565}$$

$$[g_{40}(t)+\gamma g_{41}(t)+\beta g_{42}(t)+\alpha g_{43}(t)]^m = g_{40}(mt)+\gamma g_{41}(mt)+\beta g_{42}(mt)+\alpha g_{43}(mt). \tag{566}$$

Since

$$(\alpha+\gamma)^{2m} = 2^{2m-1}(1+\beta),\ (\alpha+\gamma)^{2m+1} = 2^{2m}(\alpha+\gamma), \tag{567}$$

it can be shown from the definition of the exponential function, Eq. (44) that

$$\exp(\alpha+\gamma)x = e_1 + \frac{1+\beta}{2}\cosh 2x + \frac{\alpha+\gamma}{2}\sinh 2x. \tag{568}$$

Substituting in the relation $\exp(\alpha+\gamma)x = \exp\alpha x \exp\gamma x$ the expression of the exponentials from Eqs. (543), (544) and (568) yields

$$g_{40}^2 + g_{41}^2 + g_{42}^2 + g_{43}^2 = \frac{1+\cosh 2x}{2}, \tag{569}$$

$$g_{40}g_{42} + g_{41}g_{43} = \frac{-1+\cosh 2x}{4}, \tag{570}$$

$$g_{40}g_{41} + g_{40}g_{43} + g_{41}g_{42} + g_{42}g_{43} = \frac{1}{2}\sinh 2x, \tag{571}$$

where $g_{40}, g_{41}, g_{42}, g_{43}$ are functions of x.

Similarly, since

$$(\alpha-\gamma)^{2m} = (-1)^m 2^{2m-1}(1-\beta),\ (\alpha-\gamma)^{2m+1} = (-1)^m 2^{2m}(\alpha-\gamma), \tag{572}$$

it can be shown from the definition of the exponential function, Eq. (44) that

$$\exp(\alpha-\gamma)x = \frac{1+\beta}{2} + e_1\cos 2x + \tilde{e}_1 \sin 2x. \tag{573}$$

Substituting in the relation $\exp(\alpha-\gamma)x = \exp\alpha x \exp(-\gamma x)$ the expression of the exponentials from Eqs. (543), (544) and (573) yields

$$g_{40}^2 - g_{41}^2 + g_{42}^2 - g_{43}^2 = \frac{1+\cos 2x}{2}, \tag{574}$$

$$g_{40}g_{42} - g_{41}g_{43} = \frac{1-\cos 2x}{2}, \tag{575}$$

$$g_{40}g_{41} - g_{40}g_{43} - g_{41}g_{42} + g_{42}g_{43} = \frac{1}{2}\sin 2x, \tag{576}$$

where $g_{40}, g_{41}, g_{42}, g_{43}$ are functions of x.

The expressions of the four-dimensional cosexponential functions are

$$g_{40}(x) = \frac{1}{2}(\cosh x + \cos x), \tag{577}$$

$$g_{41}(x) = \frac{1}{2}(\sinh x + \sin x), \tag{578}$$

$$g_{42}(x) = \frac{1}{2}(\cosh x - \cos x), \tag{579}$$

$$g_{43}(x) = \frac{1}{2}(\sinh x - \sin x). \tag{580}$$

The graphs of these four-dimensional cosexponential functions are shown in Fig. 4.

It can be checked that the cosexponential functions are solutions of the fourth-order differential equation

$$\frac{\mathrm{d}^4\zeta}{\mathrm{d}u^4} = \zeta, \tag{581}$$

whose solutions are of the form $\zeta(u) = Ag_{40}(u) + Bg_{41}(u) + Cg_{42(u)} + Dg_{43}(u)$. It can also be checked that the derivatives of the cosexponential functions are related by

$$\frac{dg_{40}}{dw} = g_{43},\ \frac{dg_{41}}{dw} = g_{40},\ \frac{dg_{42}}{dw} = g_{41},\ \frac{dg_{43}}{dw} = g_{42}. \tag{582}$$

5.4 The exponential and trigonometric forms of a polar fourcomplex number

The polar fourcomplex numbers $u = x + \alpha y + \beta z + \gamma t$ for which $v_+ = x + y + z + t > 0$, $v_- = x - y + z - t > 0$ can be written in the form

$$x + \alpha y + \beta z + \gamma t = e^{x_1 + \alpha y_1 + \beta z_1 + \gamma t_1}. \tag{583}$$

The expressions of x_1, y_1, z_1, t_1 as functions of x, y, z, t can be obtained by developing $e^{\alpha y_1}, e^{\beta z_1}$ and $e^{\gamma t_1}$ with the aid of Eqs. (542)-(544), by multiplying these expressions and separating the hypercomplex components, and then substituting the expressions of the four-dimensional cosexponential functions, Eqs. (577)-(580),

$$x + y + z + t = e^{x_1 + y_1 + z_1 + t_1}, \tag{584}$$

$$x - z = e^{x_1 - z_1} \cos(y_1 - t_1), \tag{585}$$

$$x - y + z - t = e^{x_1 - y_1 + z_1 - t_1}, \tag{586}$$

$$y - t = e^{x_1 - z_1} \sin(y_1 - t_1). \tag{587}$$

It can be shown from Eqs. (584)-(587) that

$$x_1 = \frac{1}{2} \ln(\mu_+ \mu_-),\ y_1 = \frac{1}{2}(\phi + \omega),\ z_1 = \frac{1}{2} \ln \frac{\mu_-}{\mu_+},\ t_1 = \frac{1}{2}(\phi - \omega), \tag{588}$$

where

$$\mu_-^2 = (x + z)^2 - (y + t)^2 = v_+ v_-,\ \ v_+ > 0,\ \ v_- > 0. \tag{589}$$

The quantitites ϕ and ω are determined by

$$\cos \phi = (x - z)/\mu_+,\ \sin \phi = (y - t)/\mu_+ \tag{590}$$

and

$$\cosh \omega = (x + z)/\mu_-,\ \sinh \omega = (y + t)/\mu_-. \tag{591}$$

The explicit form of ω is

$$\omega = \frac{1}{2} \ln \frac{x + y + z + t}{x - y + z - t}. \tag{592}$$

If $u = u_1u_2$, and $\mu_{j-}^2 = (x_j+z_j)^2-(y_j+t_j)^2, j = 1,2$, it can be checked with the aid of Eqs. (503) that

$$\mu_- = \mu_{1-}\mu_{2-}. \tag{593}$$

Moreover, if $\cosh\omega_j = (x_j+z_j)/\mu_{j-}$, $\sinh\omega_j = (y_j+t_j)/\mu_{j-}, j = 1,2$, it can be checked that

$$\omega = \omega_1+\omega_2. \tag{594}$$

The relation (594) is a consequence of the identities

$$\begin{aligned}&(x_1x_2+z_1z_2+y_1t_2+t_1y_2)+(x_1z_2+z_1x_2+y_1y_2+t_1t_2)\\&\quad=(x_1+z_1)(x_2+z_2)+(y_1+t_1)(y_2+t_2),\end{aligned} \tag{595}$$

$$\begin{aligned}&(x_1y_2+y_1x_2+z_1t_2+t_1z_2)+(x_1t_2+t_1x_2+z_1y_2+y_1z_2)\\&\quad=(y_1+t_1)(x_2+z_2)+(x_1+z_1)(y_2+t_2).\end{aligned} \tag{596}$$

According to Eq. (590), ϕ is a cyclic variable, $0\leq\phi<2\pi$. As it has been assumed that $v_+>0, v_->0$, it follows that $x+z>0$ and $x+z>|y+t|$. The range of the variable ω is $-\infty<\omega<\infty$. The exponential form of the polar fourcomplex number u is then

$$u = \rho\exp\left[\frac{1}{2}\beta\ln\frac{\mu_-}{\mu_+}+\frac{1}{2}\alpha(\omega+\phi)+\frac{1}{2}\gamma(\omega-\phi)\right], \tag{597}$$

where

$$\rho = (\mu_+\mu_-)^{1/2}. \tag{598}$$

If $u = u_1u_2$, and $\rho_j = (\mu_{j+}\mu_{j-})^{1/2}, j = 1,2$, then from Eqs. (509) and (593) it results that

$$\rho = \rho_1\rho_2. \tag{599}$$

It can be checked with the aid of Eq. (533) that

$$\exp(\tilde{e}_1\phi) = \frac{1+\beta}{2}+e_1\cos\phi+\tilde{e}_1\sin\phi, \tag{600}$$

which shows that $e^{(\alpha-\gamma)\phi/2}$ is a periodic function of ϕ, with period 2π. The modulus has the property that

$$|u\exp(\tilde{e}_1\phi)| = |u|. \tag{601}$$

By introducing in Eq. (597) the polar angles θ_+, θ_- defined in Eqs. (515), the exponential form of the fourcomplex number u becomes

$$u = \rho \exp\left[\frac{1}{4}(\alpha+\beta+\gamma)\ln\frac{\sqrt{2}}{\tan\theta_+} - \frac{1}{4}(\alpha-\beta+\gamma)\ln\frac{\sqrt{2}}{\tan\theta_-} + \tilde{e}_1\phi\right], \tag{602}$$

where $0 < \theta_+ < \pi/2, 0 < \theta_- < \pi/2$. The relation between the amplitude ρ, Eq. (598), and the distance d, Eq. (501), is, according to Eqs. (515) and (516),

$$\rho = \frac{2^{3/4}d}{(\tan\theta_+\tan\theta_-)^{1/4}}\left(1+\frac{1}{\tan^2\theta_+}+\frac{1}{\tan^2\theta_-}\right)^{-1/2}. \tag{603}$$

Using the properties of the vectors e_+, e_- written in Eq. (531), the first part of the exponential, Eq. (602) can be developed as

$$\exp\left[\frac{1}{4}(\alpha+\beta+\gamma)\ln\frac{\sqrt{2}}{\tan\theta_+} - \frac{1}{4}(\alpha-\beta+\gamma)\ln\frac{\sqrt{2}}{\tan\theta_-}\right]$$
$$= \left(\frac{1}{2}\tan\theta_+\tan\theta_-\right)^{1/4}\left(e_1 + e_+\frac{\sqrt{2}}{\tan\theta_+} + e_-\frac{\sqrt{2}}{\tan\theta_-}\right). \tag{604}$$

The fourcomplex number u, Eq. (602), can then be written as

$$u = d\sqrt{2}\left(1+\frac{1}{\tan^2\theta_+}+\frac{1}{\tan^2\theta_-}\right)^{-1/2}\left(e_1 + e_+\frac{\sqrt{2}}{\tan\theta_+} + e_-\frac{\sqrt{2}}{\tan\theta_-}\right)\exp\left(\tilde{e}_1\phi\right), \tag{605}$$

which is the trigonometric form of the fourcomplex number u.

The polar angles θ_+, θ_-, Eq. (515), can be expressed in terms of the variables θ, λ, Eq. (523), as

$$\tan\theta_+ = \frac{1}{\tan\theta\cos\lambda},\ \tan\theta_- = \frac{1}{\tan\theta\sin\lambda}, \tag{606}$$

so that

$$1+\frac{1}{\tan^2\theta_+}+\frac{1}{\tan^2\theta_-} = \frac{1}{\cos^2\theta}. \tag{607}$$

The exponential form of the fourcomplex number u, written in terms of the amplitude ρ and of the angles θ, λ, ϕ is

$$u = \rho\exp\left[\frac{1}{4}(\alpha+\beta+\gamma)\ln(\sqrt{2}\tan\theta\cos\lambda) - \frac{1}{4}(\alpha-\beta+\gamma)\ln(\sqrt{2}\tan\theta\sin\lambda) + \tilde{e}_1\phi\right], \tag{608}$$

where $0 \le \lambda < \pi/2$. The trigonometric form of the fourcomplex number u, written in terms of the amplitude ρ and of the angles θ, λ, ϕ is

$$u = d\sqrt{2}\left(e_1\cos\theta + e_+\sqrt{2}\sin\theta\cos\lambda + e_-\sqrt{2}\sin\theta\sin\lambda\right)\exp\left(\tilde{e}_1\phi\right). \tag{609}$$

If $u = u_1u_2$, it can be shown with the aid of the trigonometric form, Eq. (605), that the modulus of the product as a function of the polar angles is

$$d^2 = 4d_1^2d_2^2\left(\frac{1}{2}+\frac{1}{\tan^2\theta_{1+}\tan^2\theta_{2+}}+\frac{1}{\tan^2\theta_{1-}\tan^2\theta_{2-}}\right)$$
$$\left(1+\frac{1}{\tan^2\theta_{1+}}+\frac{1}{\tan^2\theta_{1-}}\right)^{-1}\left(1+\frac{1}{\tan^2\theta_{2+}}+\frac{1}{\tan^2\theta_{2-}}\right)^{-1}. \qquad (610)$$

The modulus d of the product u_1u_2 can be expressed alternatively in terms of the angles θ, λ, ϕ with the aid of the trigonometric form, Eq. (609), as

$$d^2 = 4d_1^2d_2^2\left(\frac{1}{2}\cos^2\theta_1\cos^2\theta_2+\sin^2\theta_1\sin^2\theta_2\cos^2\lambda_1\cos^2\lambda_2+\sin^2\theta_1\sin^2\theta_2\sin^2\lambda_1\sin^2\lambda_2\right). \qquad (611)$$

5.5 Elementary functions of polar fourcomplex variables

The logarithm u_1 of the polar fourcomplex number u, $u_1 = \ln u$, can be defined as the solution of the equation

$$u = e^{u_1}, \qquad (612)$$

written explicitly previously in Eq. (583), for u_1 as a function of u. From Eq. (597) it results that

$$\ln u = \ln\rho + \frac{1}{2}\beta\ln\frac{\mu_-}{\mu_+} + \frac{1}{2}\alpha(\omega+\phi) + \frac{1}{2}\gamma(\omega-\phi). \qquad (613)$$

If the fourcomplex number u is written in terms of the amplitude ρ and of the angles θ_+, θ_-, ϕ, the logarithm is

$$\ln u = \ln\rho + \frac{1}{4}(\alpha+\beta+\gamma)\ln\frac{\sqrt{2}}{\tan\theta_+} - \frac{1}{4}(\alpha-\beta+\gamma)\ln\frac{\sqrt{2}}{\tan\theta_-} + \tilde{e}_1\phi, \qquad (614)$$

where $0 < \theta_+ < \pi/2, 0 < \theta_+ < \pi/2$. If the fourcomplex number u is written in terms of the amplitude ρ and of the angles θ, λ, ϕ, the logarithm is

$$\ln u = \ln\rho + \frac{1}{4}(\alpha+\beta+\gamma)\ln(\sqrt{2}\tan\theta\cos\lambda) - \frac{1}{4}(\alpha-\beta+\gamma)\ln(\sqrt{2}\tan\theta\sin\lambda) + \tilde{e}_1\phi, \qquad (615)$$

where $0 < \theta < \pi/2, 0 \le \lambda < \pi/2$. The logarithm is multivalued because of the term proportional to ϕ. It can be inferred from Eq. (614) that

$$\ln(u_1u_2) = \ln u_1 + \ln u_2, \qquad (616)$$

up to multiples of $\pi(\alpha-\gamma)$. If the expressions of ρ, μ_+, μ_- and ω in terms of x, y, z, t are introduced in Eq. (613), the logarithm of the polar fourcomplex number becomes

$$\ln u = \frac{1+\alpha+\beta+\gamma}{4}\ln(x+y+z+t)+\frac{1-\alpha+\beta-\gamma}{4}\ln(x-y+z-t)+e_1\ln\mu_+ +\tilde{e}_1\phi. \tag{617}$$

The power function u^m can be defined for $v_+>0, v_->0$ and real values of m as

$$u^m = e^{m\ln u}. \tag{618}$$

The power function is multivalued unless m is an integer. For integer m, it can be inferred from Eqs. (613) and (618) that

$$(u_1u_2)^m = u_1^m\, u_2^m. \tag{619}$$

Using the expression (617) for $\ln u$ and the relations (531)-(534) it can be shown that

$$u^m = e_+v_+^m + e_-v_-^m + \mu_+^m\left(e_1\cos m\phi + \tilde{e}_1\sin m\phi\right). \tag{620}$$

For integer m, the relation (620) is valid for any x, y, z, t. For natural m this relation can be written as

$$u^m = e_+v_+^m + e_-v_-^m + \left[e_1(x-z)+\tilde{e}_1(y-t)\right]^m, \tag{621}$$

as can be shown with the aid of the relation

$$e_1\cos m\phi + \tilde{e}_1\sin m\phi = \left(e_1\cos\phi + \tilde{e}_1\sin\phi\right)^m, \tag{622}$$

valid for natural m. For $m=-1$ the relation (620) becomes

$$\begin{aligned}\frac{1}{x+\alpha y+\beta z+\gamma t} &= \frac{1}{4}\left(\frac{1+\alpha+\beta+\gamma}{x+y+z+t}+\frac{1-\alpha+\beta-\gamma}{x-y+z-t}\right)\\ &+\frac{1}{2}\frac{(1-\beta)(x-z)-(\alpha-\gamma)(y-t)}{(x-z)^2+(y-t)^2}.\end{aligned} \tag{623}$$

If $m=2$, it can be checked that the right-hand side of Eq. (621) is equal to $(x+\alpha y+\beta z+\gamma t)^2 = x^2+z^2+2yt+2\alpha(xy+zt)+\beta(y^2+t^2+2xz)+2\gamma(xt+yz)$.

The trigonometric functions of the fourcomplex variable u and the addition theorems for these functions have been written in Eqs. (90)-(93). The cosine and sine functions of the hypercomplex variables $\alpha y, \beta z$ and γt can be expressed as

$$\cos\alpha y = g_{40} - \beta g_{42},\ \sin\alpha y = \alpha g_{41} - \gamma g_{43}, \tag{624}$$

$$\cos\beta y = \cos y,\ \sin\beta y = \beta\sin y, \tag{625}$$

$$\cos\gamma y = g_{40} - \beta g_{42},\ \sin\gamma y = \gamma g_{41} - \alpha g_{43}. \tag{626}$$

The cosine and sine functions of a polar fourcomplex number $x + \alpha y + \beta z + \gamma t$ can then be expressed in terms of elementary functions with the aid of the addition theorems Eqs. (92), (93) and of the expressions in Eqs. (624)-(626).

The hyperbolic functions of the fourcomplex variable u and the addition theorems for these functions have been written in Eqs. (97)-(100). The hyperbolic cosine and sine functions of the hypercomplex variables $\alpha y, \beta z$ and γt can be expressed as

$$\cosh\alpha y = g_{40} + \beta g_{42},\ \sinh\alpha y = \alpha g_{41} + \gamma g_{43}, \tag{627}$$

$$\cosh\beta y = \cosh y,\ \sinh\beta y = \beta\sinh y, \tag{628}$$

$$\cosh\gamma y = g_{40} + \beta g_{42},\ \sinh\gamma y = \gamma g_{41} + \alpha g_{43}. \tag{629}$$

The hyperbolic cosine and sine functions of a polar fourcomplex number $x + \alpha y + \beta z + \gamma t$ can then be expressed in terms of elementary functions with the aid of the addition theorems Eqs. (99), (100) and of the expressions in Eqs. (627)-(629).

5.6 Power series of polar fourcomplex variables

A polar fourcomplex series is an infinite sum of the form

$$a_0 + a_1 + a_2 + \cdots + a_l + \cdots, \tag{630}$$

where the coefficients a_l are polar fourcomplex numbers. The convergence of the series (630) can be defined in terms of the convergence of its 4 real components. The convergence of a polar fourcomplex series can however be studied using polar fourcomplex variables. The main criterion for absolute convergence remains the comparison theorem, but this requires a number of inequalities which will be discussed further.

The modulus of a polar fourcomplex number $u = x + \alpha y + \beta z + \gamma t$ can be defined as

$$|u| = (x^2 + y^2 + z^2 + t^2)^{1/2}, \tag{631}$$

so that according to Eq. (501) $d=|u|$. Since $|x|\leq|u|, |y|\leq|u|, |z|\leq|u|, |t|\leq|u|$, a property of absolute convergence established via a comparison theorem based on the modulus of the series (630) will ensure the absolute convergence of each real component of that series.

The modulus of the sum u_1+u_2 of the polar fourcomplex numbers u_1, u_2 fulfils the inequality

$$||u_1|-|u_2||\leq|u_1+u_2|\leq|u_1|+|u_2|. \tag{632}$$

For the product the relation is

$$|u_1u_2|\leq 2|u_1||u_2|, \tag{633}$$

which replaces the relation of equality extant for regular complex numbers. The equality in Eq. (633) takes place for $x_1=y_1=z_1=t_1, x_2=y_2=z_2=t_2$, or $x_1=-y_1=z_1=-t_1, x_2=-y_2=z_2=-t_2$. In particular

$$|u^2|\leq 2(x^2+y^2+z^2+t^2). \tag{634}$$

The inequality in Eq. (633) implies that

$$|u^l|\leq 2^{l-1}|u|^l. \tag{635}$$

From Eqs. (633) and (635) it results that

$$|au^l|\leq 2^l|a||u|^l. \tag{636}$$

A power series of the polar fourcomplex variable u is a series of the form

$$a_0+a_1u+a_2u^2+\cdots+a_lu^l+\cdots. \tag{637}$$

Since

$$\left|\sum_{l=0}^{\infty}a_lu^l\right|\leq\sum_{l=0}^{\infty}2^l|a_l||u|^l, \tag{638}$$

a sufficient condition for the absolute convergence of this series is that

$$\lim_{l\rightarrow\infty}\frac{2|a_{l+1}||u|}{|a_l|}<1. \tag{639}$$

Thus the series is absolutely convergent for

$$|u|<c_0, \tag{640}$$

where

$$c_0 = \lim_{l\to\infty} \frac{|a_l|}{2|a_{l+1}|}. \tag{641}$$

The convergence of the series (637) can be also studied with the aid of the formula (621) which, for integer values of l, is valid for any x, y, z, t. If $a_l = a_{lx} + \alpha a_{ly} + \beta a_{lz} + \gamma a_{lt}$, and

$$\begin{aligned}
A_{l+} &= a_{lx} + a_{ly} + a_{lz} + a_{lt},\\
A_{l-} &= a_{lx} - a_{ly} + a_{lz} - a_{lt},\\
A_{l1} &= a_{lx} - a_{lz},\\
\tilde{A}_{l1} &= a_{ly} - a_{lt},
\end{aligned} \tag{642}$$

it can be shown with the aid of relations (531)-(534) and (621) that the expression of the series (637) is

$$\sum_{l=0}^{\infty} \left[A_{l+}v_+^l e_+ + A_{l-}v_- e_- + \left(e_1 A_{l1} + \tilde{e}_1\tilde{A}_{l1}\right)(e_1 v_1 + \tilde{e}_1\tilde{v}_1)^l\right], \tag{643}$$

where the quantities v_+, v_- have been defined in Eq. (495), and the quantities $v_1, \tilde{v}_1$ have been defined in Eq. (506).

The sufficient conditions for the absolute convergence of the series in Eq. (643) are that

$$\lim_{l\to\infty} \frac{|A_{l+1,+}||v_+|}{|A_{l+}|} < 1, \lim_{l\to\infty} \frac{|A_{l+1,-}||v_-|}{|A_{l-}|} < 1, \lim_{l\to\infty} \frac{A_{l+1}\mu_+}{A_l} < 1, \tag{644}$$

where the real and positive quantity $A_{l-} > 0$ is given by

$$A_l^2 = A_{l1}^2 + \tilde{A}_{l1}^2. \tag{645}$$

Thus the series in Eq. (643) is absolutely convergent for

$$|x+y+z+t| < c_+,\ |x-y+z-t| < c_-,\ \mu_+ < c_1 \tag{646}$$

where

$$c_+ = \lim_{l\to\infty} \frac{|A_{l+}|}{|A_{l+1,+}|},\ c_- = \lim_{l\to\infty} \frac{|A_{l-}|}{|A_{l+1,-}|},\ c_1 = \lim_{l\to\infty} \frac{A_{l-}}{A_{l+1,-}}. \tag{647}$$

The relations (646) show that the region of convergence of the series (643) is a four-dimensional cylinder. It can be shown that $c_0 = (1/2)\min(c_+, c_-, c_1)$, where min designates the smallest of the numbers in the argument of this function. Using the expression of $|u|$ in Eq. (508), it can be seen that the spherical region of convergence defined in Eqs. (640), (641) is included in the cylindrical region of convergence defined in Eqs. (647).

5.7 Analytic functions of polar fourcomplex variables

The fourcomplex function $f(u)$ of the fourcomplex variable u has been expressed in Eq. (136) in terms of the real functions $P(x,y,z,t), Q(x,y,z,t), R(x,y,z,t), S(x,y,z,t)$ of real variables x,y,z,t. The relations between the partial derivatives of the functions P,Q,R,S are obtained by setting succesively in Eq. (137) $\Delta x \to 0, \Delta y = \Delta z = \Delta t = 0$; then $\Delta y \to 0, \Delta x = \Delta z = \Delta t = 0$; then $\Delta z \to 0, \Delta x = \Delta y = \Delta t = 0$; and finally $\Delta t \to 0, \Delta x = \Delta y = \Delta z = 0$. The relations are

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} = \frac{\partial R}{\partial z} = \frac{\partial S}{\partial t}, \tag{648}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial R}{\partial y} = \frac{\partial S}{\partial z} = \frac{\partial P}{\partial t}, \tag{649}$$

$$\frac{\partial R}{\partial x} = \frac{\partial S}{\partial y} = \frac{\partial P}{\partial z} = \frac{\partial Q}{\partial t}, \tag{650}$$

$$\frac{\partial S}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial t}. \tag{651}$$

The relations (648)-(651) are analogous to the Riemann relations for the real and imaginary components of a complex function. It can be shown from Eqs. (648)-(651) that the component P is a solution of the equations

$$\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial z^2} = 0, \quad \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial t^2} = 0, \tag{652}$$

and the components Q,R,S are solutions of similar equations. As can be seen from Eqs. (652)-(652), the components P,Q,R,S of an analytic function of polar fourcomplex variable are harmonic with respect to the pairs of variables x,y and z,t. The component P is also a solution of the mixed-derivative equations

$$\frac{\partial^2 P}{\partial x^2} = \frac{\partial^2 P}{\partial y \partial t}, \quad \frac{\partial^2 P}{\partial y^2} = \frac{\partial^2 P}{\partial x \partial z}, \quad \frac{\partial^2 P}{\partial z^2} = \frac{\partial^2 P}{\partial y \partial t}, \quad \frac{\partial^2 P}{\partial t^2} = \frac{\partial^2 P}{\partial x \partial z}, \tag{653}$$

and the components Q,R,S are solutions of similar equations. The component P is also a solution of the mixed-derivative equations

$$\frac{\partial^2 P}{\partial x \partial y} = \frac{\partial^2 P}{\partial z \partial t}, \quad \frac{\partial^2 P}{\partial x \partial t} = \frac{\partial^2 P}{\partial y \partial z}, \tag{654}$$

and the components Q,R,S are solutions of similar equations.

5.8 Integrals of functions of polar fourcomplex variables

The singularities of polar fourcomplex functions arise from terms of the form $1/(u-u_0)^m$, with $m>0$. Functions containing such terms are singular not only at $u=u_0$, but also at all points of the two-dimensional hyperplanes passing through u_0 and which are parallel to the nodal hyperplanes.

The integral of a polar fourcomplex function between two points A, B along a path situated in a region free of singularities is independent of path, which means that the integral of an analytic function along a loop situated in a region free from singularities is zero,

$$\oint_\Gamma f(u)du = 0, \tag{655}$$

where it is supposed that a surface Σ spanning the closed loop Γ is not intersected by any of the hyperplanes associated with the singularities of the function $f(u)$. Using the expression, Eq. (136) for $f(u)$ and the fact that $du = dx + \alpha dy + \beta dz + \gamma dt$, the explicit form of the integral in Eq. (655) is

$$\begin{aligned}\oint_\Gamma f(u)du = \oint_\Gamma &[(Pdx+Sdy+Rdz+Qdt)+\alpha(Qdx+Pdy+Sdz+Rdt)\\ &+\beta(Rdx+Qdy+Pdz+Sdt)+\gamma(Sdx+Rdy+Qdz+Pdt)].\end{aligned} \tag{656}$$

If the functions P, Q, R, S are regular on a surface Σ spanning the loop Γ, the integral along the loop Γ can be transformed with the aid of the theorem of Stokes in an integral over the surface Σ of terms of the form $\partial P/\partial y - \partial S/\partial x$, $\partial P/\partial z - \partial R/\partial x$, $\partial P/\partial t - \partial Q/\partial x$, $\partial R/\partial y - \partial S/\partial z$, $\partial S/\partial t - \partial Q/\partial y$, $\partial R/\partial t - \partial Q/\partial z$ and of similar terms arising from the α, β and γ components, which are equal to zero by Eqs. (648)-(651), and this proves Eq. (655).

The integral of the function $(u-u_0)^m$ on a closed loop Γ is equal to zero for m a positive or negative integer not equal to -1,

$$\oint_\Gamma (u-u_0)^m du = 0, \;\; m \text{ integer}, \; m \neq -1. \tag{657}$$

This is due to the fact that $\int (u-u_0)^m du = (u-u_0)^{m+1}/(m+1)$, and to the fact that the function $(u-u_0)^{m+1}$ is singlevalued for m an integer.

The integral $\oint_\Gamma du/(u-u_0)$ can be calculated using the exponential form (602),

$$u-u_0 = \rho \exp\left[\frac{1}{4}(\alpha+\beta+\gamma)\ln\frac{\sqrt{2}}{\tan\theta_+} - \frac{1}{4}(\alpha-\beta+\gamma)\ln\frac{\sqrt{2}}{\tan\theta_-} + \tilde{e}_1\phi\right], \tag{658}$$

so that

$$\frac{du}{u-u_0}=\frac{d\rho}{\rho}+\frac{1}{4}(\alpha+\beta+\gamma)d\ln\frac{\sqrt{2}}{\tan\theta_+}-\frac{1}{4}(\alpha-\beta+\gamma)d\ln\frac{\sqrt{2}}{\tan\theta_-}+\tilde{e}_1 d\phi. \tag{659}$$

Since ρ, $\tan\theta_+$ and $\tan\theta_-$ are singlevalued variables, it follows that $\oint_\Gamma d\rho/\rho=0$, $\oint_\Gamma d\ln\sqrt{2}/\tan\theta_+=0$, and $\oint_\Gamma d\ln\sqrt{2}/\tan\theta_+=0$. On the other hand, ϕ is a cyclic variables, so that it may give a contribution to the integral around the closed loop Γ. The result of the integrations will be given in the rotated system of coordinates

$$\xi=\frac{1}{\sqrt{2}}(x-z),\ \upsilon=\frac{1}{\sqrt{2}}(y-t),\ \tau=\frac{1}{2}(x+y+z+t),\ \upsilon=\frac{1}{2}(x-y+z-t). \tag{660}$$

Thus, if $C_\parallel$ is a circle of radius r parallel to the $\xi O\upsilon$ plane, and the projection of the center of this circle on the $\xi O\upsilon$ plane coincides with the projection of the point u_0 on this plane, the points of the circle $C_\parallel$ are described according to Eqs. (506)-(507) by the equations

$$\xi=\xi_0+r\cos\phi,\ \upsilon=\upsilon_0+r\sin\phi,\ \tau=\tau_0,\ \zeta=\zeta_0, \tag{661}$$

where $u_0=x_0+\alpha y_0+\beta z_0+\gamma t_0$, and $\xi_0,\upsilon_0,\tau_0,\zeta_0$ are calculated from x_0,y_0,z_0,t_0 according to Eqs. (660).

Then

$$\oint_{C_\parallel}\frac{du}{u-u_0}=2\pi\tilde{e}_1. \tag{662}$$

The expression of $\oint_\Gamma du/(u-u_0)$ can be written with the aid of the functional $\mathrm{int}(M,C)$ defined in Eq. (153) as

$$\oint_\Gamma\frac{du}{u-u_0}=2\pi\tilde{e}_1\ \mathrm{int}(u_{0\xi\upsilon},\Gamma_{\xi\upsilon}), \tag{663}$$

where $u_{0\xi\upsilon}$ and $\Gamma_{\xi\upsilon}$ are respectively the projections of the point u_0 and of the loop Γ on the plane $\xi\upsilon$.

If $f(u)$ is an analytic polar fourcomplex function which can be expanded in a series as written in Eq. (131), and the expansion holds on the curve Γ and on a surface spanning Γ, then from Eqs. (657) and (663) it follows that

$$\oint_\Gamma\frac{f(u)du}{u-u_0}=2\pi\tilde{e}_1\ \mathrm{int}(u_{0\xi\upsilon},\Gamma_{\xi\upsilon})f(u_0), \tag{664}$$

where $\Gamma_{\xi\upsilon}$ is the projection of the curve Γ on the plane $\xi\upsilon$, as shown in Fig. 5. Substituting in the right-hand side of Eq. (664) the expression of $f(u)$ in terms of the real components P, Q, R, S, Eq. (136), yields

$$\oint_\Gamma \frac{f(u)du}{u-u_0} = \pi\left[(\beta-1)(Q-S)+(\alpha-\gamma)(P-R)\right]\mathrm{int}\left(u_{0\xi\upsilon}, \Gamma_{\xi\upsilon}\right), \tag{665}$$

where P, Q, R, S are the values of the components of f at $u = u_0$. If the integral is written as

$$\oint_\Gamma \frac{f(u)du}{u-u_0} = I + \alpha I_\alpha + \beta I_\beta + \gamma I_\gamma, \tag{666}$$

it results from Eq. (665) that

$$I + I_\alpha + I_\beta + I_\gamma = 0. \tag{667}$$

If $f(u)$ can be expanded as written in Eq. (131) on Γ and on a surface spanning Γ, then from Eqs. (657) and (663) it also results that

$$\oint_\Gamma \frac{f(u)du}{(u-u_0)^{m+1}} = 2\frac{\pi}{m!}\tilde{e}_1\ \mathrm{int}(u_{0\xi\upsilon}, \Gamma_{\xi\upsilon})\ f^{(m)}(u_0), \tag{668}$$

where the fact has been used that the derivative $f^{(m)}(u_0)$ of order m of $f(u)$ at $u = u_0$ is related to the expansion coefficient in Eq. (131) according to Eq. (135).

If a function $f(u)$ is expanded in positive and negative powers of $u-u_j$, where u_j are polar fourcomplex constants, j being an index, the integral of f on a closed loop Γ is determined by the terms in the expansion of f which are of the form $a_j/(u-u_j)$,

$$f(u) = \cdots + \sum_j \frac{a_j}{u-u_j} + \cdots \tag{669}$$

Then the integral of f on a closed loop Γ is

$$\oint_\Gamma f(u)du = 2\pi\tilde{e}_1 \sum_j \mathrm{int}(u_{j\xi\upsilon}, \Gamma_{\xi\upsilon})a_j. \tag{670}$$

5.9 Factorization of polar fourcomplex polynomials

A polynomial of degree m of the polar fourcomplex variable $u = x + \alpha y + \beta z + \gamma t$ has the form

$$P_m(u) = u^m + a_1 u^{m-1} + \cdots + a_{m-1}u + a_m, \tag{671}$$

where the constants are in general polar fourcomplex numbers.

If $a_m = a_{mx} + \alpha a_{my} + \beta a_{mz} + \gamma a_{mt}$, and with the notations of Eqs. (495) and (642) applied for $0, 1, \cdots, m$, the polynomial $P_m(u)$ can be written as

$$P_m = \left[v_+^m + A_1 v_+^{m-1} + \cdots + A_{m-1}v_+ + A_m\right]e_+ + \left[v_-^m + A_1'' v_-^{m-1} + \cdots + A_{m-1}'' v_- + A_m''\right]e_- + \left[(e_1 v_1 + \tilde{e}_1\tilde{v}_1)^m + \sum_{l=1}^{m}\left(e_1 A_{l1} + \tilde{e}_1\tilde{A}_{l1}\right)(e_1 v_1 + \tilde{e}_1\tilde{v}_1)^{m-l}\right], \tag{672}$$

where the constants $A_{l+}, A_{l-}, A_{l1}, \tilde{A}_{l1}$ are real numbers. The polynomial of degree m in $(e_1 v_1 + \tilde{e}_1\tilde{v}_1)$ can always be written as a product of linear factors of the form $[e_1(v_1 - v_{1l}) + \tilde{e}_1(\tilde{v}_1 - \tilde{v}_{1l})]$, where the constants $v_{1l}, \tilde{v}_{1l}$ are real. The two polynomials of degree m with real coefficients in Eq. (672) which are multiplied by e_+ and e_- can be written as a product of linear or quadratic factors with real coefficients, or as a product of linear factors which, if imaginary, appear always in complex conjugate pairs. Using the latter form for the simplicity of notations, the polynomial P_m can be written as

$$P_m = \prod_{l=1}^{m}(v_+ - s_{l+})e_+ + \prod_{l=1}^{m}(v_- - s_{l-})e_- + \prod_{l=1}^{m}\left[e_1(v_1 - v_{1l}) + \tilde{e}_1(\tilde{v}_1 - \tilde{v}_{1l})\right], \tag{673}$$

where the quantities s_{l+} appear always in complex conjugate pairs, and the same is true for the quantities s_{l-}. Due to the properties in Eqs. (531)-(534), the polynomial $P_m(u)$ can be written as a product of factors of the form

$$P_m(u) = \prod_{l=1}^{m}\left[(v_+ - s_{l+})e_+ + (v_- - s_{l-})\,e_- + (e_1(v_1 - v_{1l}) + \tilde{e}_1(\tilde{v}_1 - \tilde{v}_{1l}))\right]. \tag{674}$$

These relations can be written with the aid of Eq. (535) as

$$P_m(u) = \prod_{p=1}^{m}(u - u_p), \tag{675}$$

where

$$u_p = s_{p+}e_+ + s_{p-}e_- + e_1 v_{1p} + \tilde{e}_1\tilde{v}_{1p}, p = 1, ..., m. \tag{676}$$

The roots $s_{p+}, s_{p-}, v_{1p}e_1 + \tilde{v}_{1p}\tilde{e}_1$ of the corresponding polynomials in Eq. (673) may be ordered arbitrarily. This means that Eq. (676) gives sets of m roots $u_1, ..., u_m$ of the polynomial $P_m(u)$, corresponding to the various ways in which the roots $s_{p+}, s_{p-}, v_{1p}e_1 + \tilde{v}_{1p}\tilde{e}_1$ are ordered according to p in each group. Thus, while the hypercomplex components in

Eq. (672) taken separately have unique factorizations, the polynomial $P_m(u)$ can be written in many different ways as a product of linear factors. The result of the polar fourcomplex integration, Eq. (670), is however unique.

If $P(u) = u^2 - 1$, the factorization in Eq. (675) is $u^2 - 1 = (u - u_1)(u - u_2)$, where $u_1 = \pm e_+ \pm e_- \pm e_1, u_2 = -u_1$, so that there are 4 distinct factorizations of $u^2 - 1$,

$$
\begin{aligned}
&u^2 - 1 = (u - 1)(u + 1),\\
&u^2 - 1 = (u - \beta)(u + \beta),\\
&u^2 - 1 = \left(u - \tfrac{1+\alpha-\beta+\gamma}{2}\right)\left(u + \tfrac{1+\alpha-\beta+\gamma}{2}\right),\\
&u^2 - 1 = \left(u - \tfrac{-1+\alpha+\beta+\gamma}{2}\right)\left(u + \tfrac{-1+\alpha+\beta+\gamma}{2}\right).
\end{aligned}
\tag{677}
$$

It can be checked that $\{\pm e_+ \pm e_- \pm e_1\}^2 = e_+ + e_- + e_1 = 1$.

5.10 Representation of polar fourcomplex numbers by irreducible matrices

If T is the unitary matrix,

$$
T = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0\\
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix},
\tag{678}
$$

it can be shown that the matrix TUT^{-1} has the form

$$
TUT^{-1} = \begin{pmatrix}
x+y+z+t & 0 & 0\\
0 & x-y+z-t & 0\\
0 & 0 & V_1
\end{pmatrix},
\tag{679}
$$

where U is the matrix in Eq. (533) used to represent the polar fourcomplex number u. In Eq. (679), V_1 is the matrix

$$
V_1 = \begin{pmatrix}
x-z & y-t\\
-y+t & x-z
\end{pmatrix}.
\tag{680}
$$

The relations between the variables $x+y+z+t, x-y+z-t, x-z, y-t$ for the multiplication of polar fourcomplex numbers have been written in Eqs. (504),(505), (512), (513). The matrix TUT^{-1} provides an irreducible representation [8] of the polar fourcomplex number u in terms of matrices with real coefficients.

6 Conclusions

In the case of the circular fourcomplex numbers, the operations of addition and multiplication have a simple geometric interpretation based on the amplitudes ρ, the azimuthal angles ϕ, χ, and the planar angles ψ. Exponential form exist for the circular fourcomplex numbers, involving the variables υ, ϕ, χ and ψ. The circular fourcomplex functions defined by series of powers are analytic, and the partial derivatives of the components of the circular fourcomplex functions are closely related. The integrals of circular fourcomplex functions are independent of path in regions where the functions are regular. The fact that the exponential forms of the circular fourcomplex numbers depend on the cyclic variables ϕ, χ leads to the concept of pole and residue for integrals on closed paths. The polynomials of circular fourcomplex variables can always be written as products of linear factors.

In the case of the hyperbolic fourcomplex numbers, an exponential form exists for the hyperbolic fourcomplex numbers, involving the amplitude μ and the arguments y_1, z_1, t_1. The hyperbolic fourcomplex functions defined by series of powers are analytic, and the partial derivatives of the components of the hyperbolic fourcomplex functions are closely related. The integrals of hyperbolic fourcomplex functions are independent of path in regions where the functions are regular. The polynomials of tricomplex variables can be written as products of linear or quadratic factors.

In the case of the planar fourcomplex numbers, the operations of addition and multiplication of the planar fourcomplex numbers introduced in this work have a geometric interpretation based on the amplitude ρ, the azimuthal angles ϕ, χ, and the planar angle ψ. An exponential form exists for the planar fourcomplex numbers, involving the variables ρ, ϕ, χ and ψ. The planar fourcomplex functions defined by series of powers are analytic, and the partial derivatives of the components of the planar fourcomplex functions are closely related. The integrals of planar fourcomplex functions are independent of path in regions where the functions are regular. The fact that the exponential form of the planar fourcomplex numbers depends on the cyclic variables ϕ, χ leads to the concept of pole and residue for integrals on closed paths. The polynomials of planar fourcomplex variables can always be written as products of linear or quadratic factors.

In the case of the polar fourcomplex numbers, the operations of addition and multipli-

cation have a geometric interpretation based on the amplitude σ, the polar angles θ_+, θ_- and the azimuthal angle ϕ. An exponential form exists for the polar fourcomplex numbers, involving the variables σ, ϕ and the angles θ_+, θ_-. The polar fourcomplex functions defined by series of powers are analytic, and the partial derivatives of the components of the polar fourcomplex functions are closely related. The integrals of polar fourcomplex functions are independent of path in regions where the functions are regular. The fact that the exponential form of the polar fourcomplex numbers depends on the cyclic variable ϕ leads to the concept of pole and residue for integrals on closed paths. The polynomials of polar fourcomplex variables can be written as products of linear or quadratic factors.

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FIGURE CAPTIONS

Fig. 1. Azimuthal angles ϕ, χ and planar angle ψ of the fourcomplex number $x + \alpha y + \beta z + \gamma t$, represented by the point A, situated at a distance d from the origin O.

Fig. 2. Integration path Γ and the pole u_0, and their projections $\Gamma_{\xi\upsilon}, \Gamma_{\tau\zeta}$ and $u_{0\xi\upsilon}, u_{0\tau\zeta}$ on the planes $\xi\upsilon$ and respectively $\tau\zeta$.

Fig. 3. The planar fourdimensional cosexponential functions $f_{40}, f_{41}, f_{42}, f_{43}$.

Fig. 4. The polar fourdimensional cosexponential functions $g_{40}, g_{41}, g_{42}, g_{43}$.

Fig. 5. Integration path Γ and the pole u_0, and their projections $\Gamma_{\xi\upsilon}$ and $u_{0\xi\upsilon}$ on the plane $\xi\upsilon$.

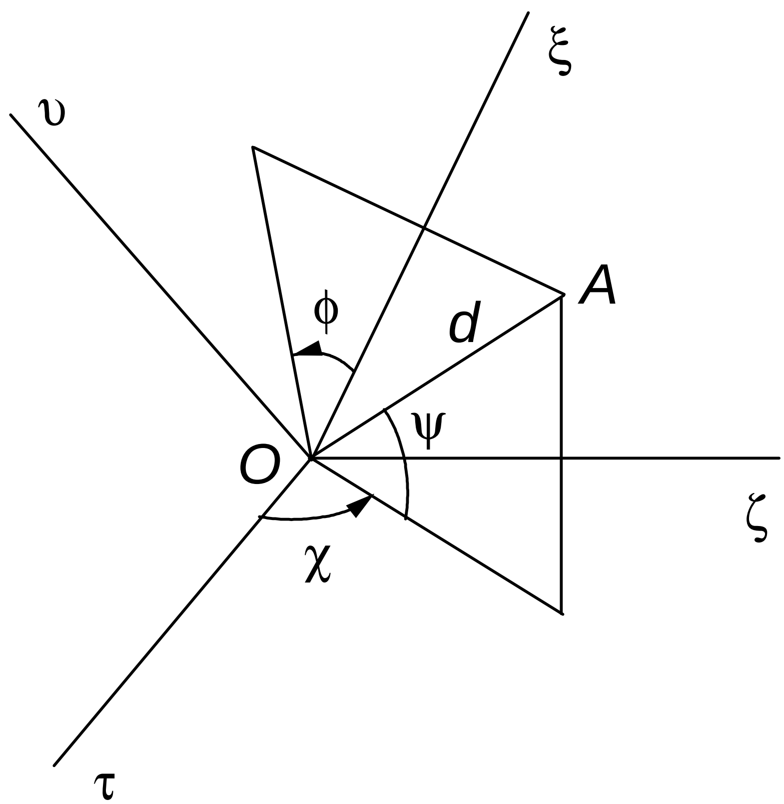
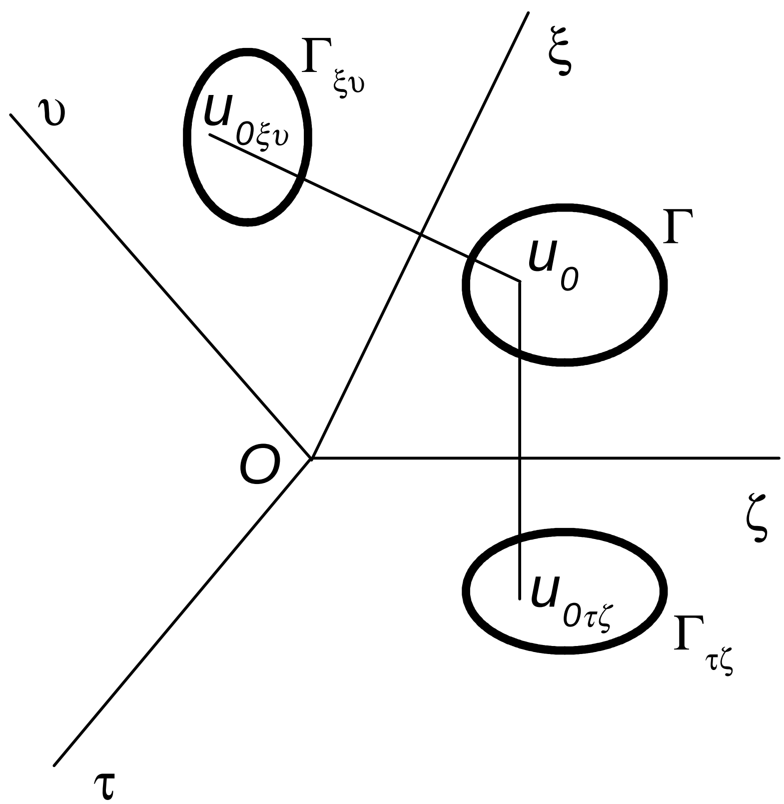


Fig. 1



$\Gamma_{\xi\upsilon}$
ξ
υ
$u_{0\xi\upsilon}$
Γ
u_0
O
ζ
$u_{0\tau\zeta}$
$\Gamma_{\tau\zeta}$
τ

Fig. 2

Fig. 3

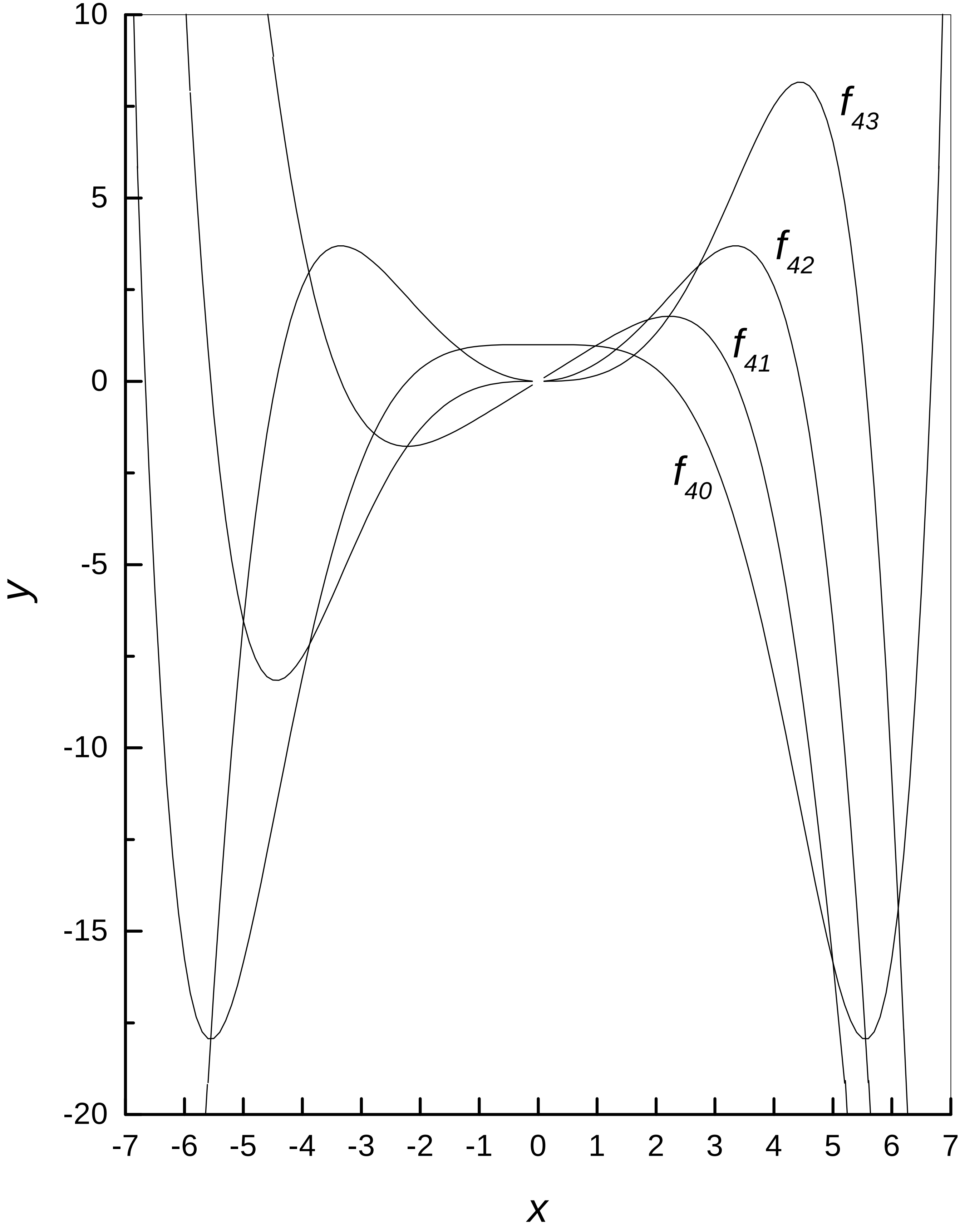
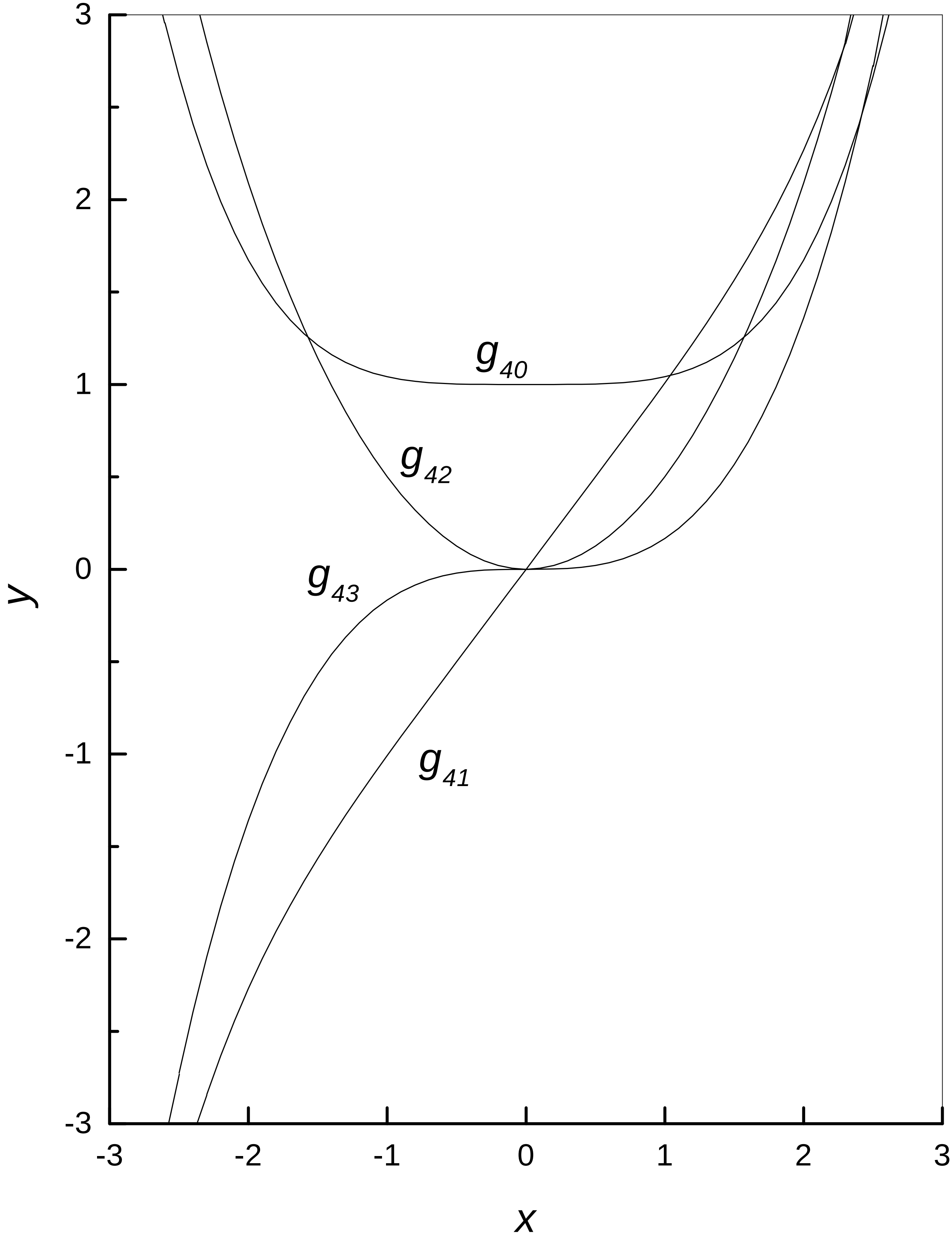


Fig. 4



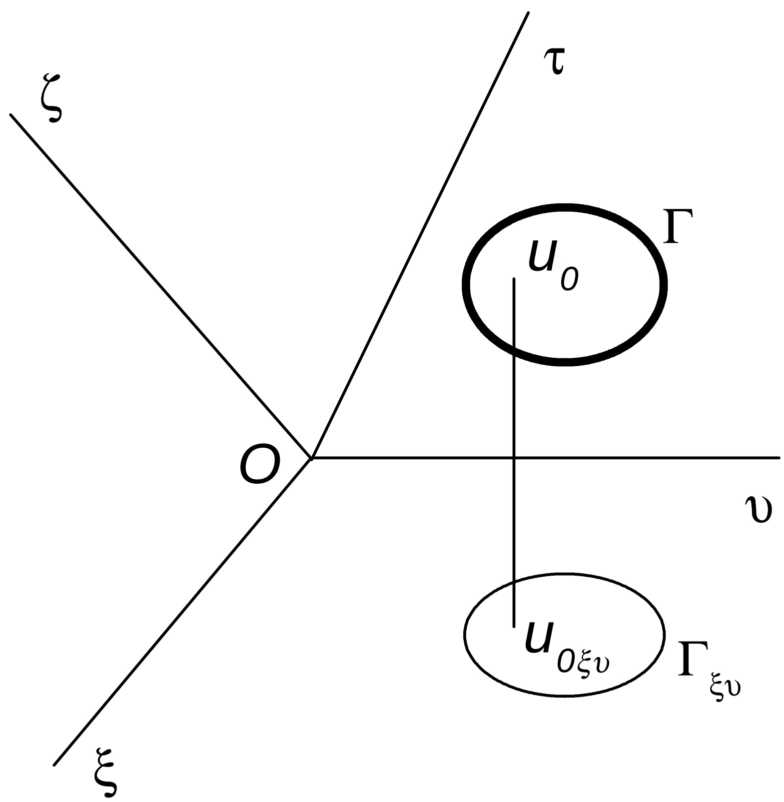


Fig. 5